Cohomology rings of arrangements

Dissertationsschrift
vorgelegt von
Carsten Schultz

Fachbereich Mathematik und Informatik
FU Berlin

Berlin, 2004

Betreuer und erster Gutachter:

Prof. Dr. Elmar Vogt
Freie Universität Berlin
Fachbereich Mathematik und Informatik
Arnimallee 14
D-14195 Berlin
vogt@math.fu-berlin.de

Zweiter Gutachter:

Prof. Dr. Günter M. Ziegler
Technische Universität Berlin
Straße des 17. Juni 136
D-10623 Berlin
ziegler@math.tu-berlin.de

Tag der Disputation: 9. Februar 2005

Contents

Introduction

This work is concerned with homotopy and homology properties of arrangements. An *arrangement* in a topological space X is a finite set $\mathcal{A}$ of subspaces of X. One goal in the study of arrangements is the description of the *union* $\bigcup \mathcal{A}$ and the *complement* $X \setminus \bigcup \mathcal{A}$. For a linear subspace arrangement in real or projective space the intersection of the union of the arrangement with the unit sphere is also of interest. It is called the *link* of the arrangement. The link can be regarded as the union of an arrangement of spheres. By Alexander duality, the homology groups of the link determine the cohomology groups of the complement of the arrangement. They do not determine the multiplication in the cohomology ring.

Main result

This work treats the general theory of arrangements as well as properties of linear arrangements. Its main contribution however is the description of the cohomology ring of the complement of a complex projective arrangement. We will describe this result here in some detail, which will give us the opportunity to introduce some terminology.

Let V be a finite dimensional vector space over $\mathbb{C}$ and let $\mathcal{A}$ be a finite set of linear subspaces of V. We denote the complex dimension of the projective space PV by n and set $P\mathcal{A} := \{PA \colon A \in \mathcal{A}\}$. We define $Q := \{\bigcap M \colon M \subset \mathcal{A}\}$ and order this set by inclusion. The partially ordered set (poset) defined in this way is called the *intersection poset* of $\mathcal{A}$. On it the *dimension function* d is defined by $d(q) := \dim Pq$. In particular $d(\top) = d(\bigcap \emptyset) = d(V) = n$. The result will be an explicit description of the cohomology ring of the complement of the projective arrangement $P\mathcal{A}$ in terms of the intersection poset and the dimension function.

Additively the cohomology of the complement is given by

$$H^{2n-i}\left(PV \setminus \bigcup P\mathcal{A}\right) \cong H_i\left(PV, \bigcup P\mathcal{A}\right) \cong \bigoplus_{k=0}^{n} H_{i-2k}(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}). \quad (1)$$

Here $Q_{[k,n]} := \{q \in Q \colon k \leq d(q) \leq n\}$, $Q_{[k,n)} := \{q \in Q \colon k \leq d(q) < n\}$ and for a poset P the *order complex of* P, i.e. the simplicial complex with simplices all chains in P, is denoted by ΔP. The first isomorphism above is Poincaré-Lefschetz-duality, and the second isomorphism will be described explicitly by maps $h_k\colon H_{i-2k}(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \to H_i\,(PV, \bigcup P\mathcal{A})$.

On the ordered simplicial chain complex C_* of ΔQ a product $\hat{\times}$ is defined as the composition

$$\hat{\times}\colon C_r(\Delta Q)\otimes C_s(\Delta Q) \xrightarrow{\times} C_{r+s}(\Delta Q\times\Delta Q) = C_{r+s}(\Delta(Q\times Q)) \xrightarrow{\wedge_*} C_{r+s}(\Delta Q), \quad (2)$$

where $\wedge\colon Q\times Q \to Q$ is the map taking (u,v) to the minimum $u\cap v$. The map $\wedge$ is order preserving and hence a simplicial map. The product $\hat{\times}$ induces products in homology. Denoting the intersection product on the homology of $(PV, \bigcup P\mathcal{A})$ that corresponds via duality to the cup product on the cohomology of the complement $PV \setminus \bigcup P\mathcal{A}$ by $\bullet$ the cohomology ring will be fully described by (1) together with the following formula.

Theorem. *For $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ and $d \in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$*

$$h_k(c) \bullet h_l(d) = \begin{cases} h_{k+l-n}(c \,\hat{\times}\, d), & k+l \geq n, \\ 0, & k+l < n. \end{cases} \quad (3)$$

This will appear as Theorem 2.2.1 in this work. Since the intersection product is a composition of the cross product and the transfer of the diagonal map, the connection between it and the product $\hat{\times}$ seems very natural in the light of (2).

History

We mention parts of the history of this subject that build a suitable context for the description of the content of this work.

Arnol'd has given a simple presentation in terms of generators and relations for the cohomology ring of the classifying space of the coloured braid group [Arn69]. This classifying space is the complement of the arrangement $\{\{z\colon z_i = z_j\} : i \neq j\}$ in $\mathbb{C}^n$, i.e. it is the complement of a linear complex hyperplane arrangement. This result has been extended to arbitrary linear complex hyperplane arrangements, where the cohomology is described by the Orlik-Solomon algebra of the intersection poset [OS80]. Several generalizations for other classes of complex linear subspace arrangements have been obtained afterwards.

For an arbitrary complex linear and projective subspace arrangement Goresky and MacPherson have given descriptions of the cohomology groups of the complement in terms of the intersection poset and the dimension function as an application of their stratified Morse theory [GM88]. A formula equivalent to (1) appears in that work. Ziegler and Živaljević have given a concrete homotopy equivalence between a space determined by the intersection poset and the dimension function and the link of a linear arrangement from which the homology formula can be read of [ZŽ93]. Their approach is to view an arrangement as a diagram of spaces (and inclusion maps). They have given an overview of homotopy theoretic tools which are useful in this setting and further applications to problems in combinatorics in [WZŽ99].

De Concini and Procesi produced rational models to show that the cohomology rings with rational coefficients of the complements of complex linear arrangements are determined by the intersection poset and the dimension function [DCP95]. Yuzvinsky used these models to endow the homology formulas given by Goresky and MacPherson for these arrangements with a combinatorially defined product which describes the cohomology ring of the complement of a complex linear arrangement [Yuz02]. This product is equivalent to the product $\hat{\times}$ defined above under an isomorphism shifting dimensions by two, although the connection with the cross product was not made explicit. Starting from this description of the cohomology ring he was in a position to attack the problem of giving presentations in terms of generators and relations for special classes of arrangements in a purely combinatorial way and he generalized previous results [Yuz99]. His results were however, by the nature of the rational models which were at the foundations of this, confined to complex arrangements and cohomology with rational coefficients. The generalization of the product formula to integral coefficients and a class of real linear arrangements containing all complex linear arrangements was done independently by Deligne, Goresky and MacPherson [DGM00] and by de Longueville and the current author [dLS01]. The latter work uses quite explicit geometrical constructions for which it is important that the homology isomorphism are induced by the topological maps of Ziegler and Živaljević. It also introduces the product $\hat{\times}$ in the form above.

Leitfaden

In Chapter 1 we deal with general arrangements in topological spaces. We first give a minimal overview over homotopy properties of diagrams of spaces. We then develop a corresponding theory of diagrams of chain complexes suitable to the study of homology properties of arrangements. This section features a spectral sequence that will be crucial in studying products later on. In the third section we show how to apply the results presented so far to the study of arrangements. Possibly new is the proof that a product formula like Yuzvinsky's holds quite generally for the cohomology ring of the complement of an almost arbitrary arrangement in a manifold, albeit only in the graded object defined by the filtration of the cohomology ring induced by the spectral sequence. This graded formula will be be the basis for proofs of exact formulas in the second chapter. In the case of projective arrangements discussed above, such a graded formula would describe $h_k(c) \bullet h_l(d)$ only up to elements $h_i(r_i)$ with $i > k + l - n$.

In Chapter 2 we enter the more concrete realm of linear arrangements. In Section 2.1 we prove several homology and also a few homotopy formulas for central linear, projective and affine arrangements, among them (1). While the isomorphisms and homotopy equivalences are constructed in a uniform manner. Still some redundancy is to be expected, as the aim is to demonstrate the use of the tools from Chapter 1 and connections between the different isomorphisms. There is probably nothing really new in that section, except perhaps that some things may not have been made quite so explicit before, including the connections between affine and projective arrangements and the homotopy equivalence for projective arrangements in Proposition 2.1.17. The topological maps forming this homotopy equivalance also induce the isomorphisms h_k defined above. To have such explicit descriptions of them makes our approach to the calculations of products possible.

In Section 2.2 we turn to determining the products in the cohomology rings of complements of linear arrangements. We first state the results for affine and projective arrangements. The product formula in Theorem 2.2.3 is the main result of [dLS01]. The corresponding formula for projective arrangements in Theorem 2.2.1 is the one discussed above. We then prove graded versions of these formulas by identifying them as special cases of a result from the first chapter. We show how the exact formulas can be derived from these by an inductive argument, if the vanishing of certain products can be guaranteed. An easy geometric argument then proves this vanishing for affine arrangements. For projective arrangements, this necessary vanishing is just the case $k + l < n$ in (3) above. Its proof costs considerably more effort than in the affine case and takes up Section 2.3.

In Section 2.4 we derive from the product formula for projective arrangements thus proved a presentation of the cohomology ring of a projective c-arrangement. This is done by methods employed by Yuzvinsky in proving presentations for similar classes of linear arrangements.

Acknowledgements

My thanks go to Elmar Vogt for supervising this thesis and expressing his confidence in me during those inevitable phases of doubt.

From Mark de Longueville I learned about linear arrangements and the questions regarding their cohomology rings. Working together with him was a most enjoyable experience. When I later took up the study of arrangements again, he was always open to discussions. Thanks for your friendship, Mark!

With Stefan Geschke I had very pleasant mathematical conversations while running around a lake near the math department that helped keeping my appreciation of mathematics as a whole alive.

Chapter 1

General Arrangements

For a topological space X and a set $\mathcal{A}$ of subspaces of X (an *arrangement* in X) we will be interested in the homotopy type of $\bigcup \mathcal{A}$ and in the calculation of the homology groups of this space. We will also be interested in the space $X \setminus \bigcup \mathcal{A}$, especially in its cohomology ring. If X is a manifold, we will describe the latter, using Poincaré duality, via the homology of the pair $(X, \bigcup \mathcal{A})$ and intersection products therein.

The arrangement $\mathcal{A}$ defines a partially ordered set (*poset*), the intersection poset $Q := \{\bigcap S \colon S \subset Q\}$ ordered by inclusion. This is a special case of a *diagram of spaces*. We will more generally investigate diagrams over a small category $\mathfrak{C}$ (instead of the poset Q) where maps are not required to be inclusions. This additional generality will not complicate the proofs, and we hope that it will clarify the relevant concepts by allowing the reader the comparison with other special cases, for example that of the category $\mathfrak{C}$ being a group. We also envision applications of these tools to arrangements with group actions where $\mathfrak{C}$ would be the orbit category of the group, but this will not be explained in the current work.

1.1 Diagrams of spaces

In this section we will denote by $\mathfrak{T}op$ a suitable category of topological spaces, such as the category presented in [Vog71].

Let $\mathfrak{C}$ be a (discrete) small category. We call an $X \in \mathfrak{T}op^{\mathfrak{C}}$, i.e. a functor $X \colon \mathfrak{C} \to \mathfrak{T}op$, a $\mathfrak{C}$-diagram of spaces. If the category $\mathfrak{C}$ is a group G, a G-diagram is a space with a G-operation defined on it. We will choose similar notation in such a way that an $X \in \mathfrak{T}op^{\mathfrak{C}}$ corresponds to a left operation and an $X \in \mathfrak{T}op^{\mathfrak{C}^o}$ corresponds to a right operation.

Diagrams of spaces have been used as a tool for studying homotopy types of arrangements in [ZŽ93] and [WZŽ99]. Since our main interest lies in computing homology groups and products, not in determining homotopy types, we summarize the needed results only briefly, giving proofs only where they illustrate concepts that will be useful later on, and using [HV92] and [FF89] as references. This is also hoped to motivate the material on diagrams of chain complexes in the next section, which proceeds analogously in some respects. In particular, we want to draw attention to the notions of free diagrams and ZŽ–maps.

The setting in which these results will be applied to arrangements is explained in Section 1.3.

1.1.1 Definition. Let $X \in \mathfrak{Top}^{\mathfrak{C}^o}$, $Y \in \mathfrak{Top}^{\mathfrak{C}}$. We define

$$X \times_{\mathfrak{C}} Y := \coprod_{q \in \operatorname{Obj} \mathfrak{C}} X(q) \times Y(q) \Big/ \sim, \tag{1.1}$$

where $\sim$ is the relation generated by $(x, Y(f)y) \sim (X(f)x, y)$ for $x \in X(q)$, $y \in Y(p)$, $f \in \mathfrak{C}(p,q)$.

1.1.2 Definition. For $S \in \mathfrak{Top}^{\operatorname{Obj}\mathfrak{C}}$ we define $i_{\#}S \in \mathfrak{Top}^{\mathfrak{C}}$ by

$$(i_{\#}S)(q) := \coprod_{p \in \operatorname{Obj}\mathfrak{C}} \coprod_{f \in \mathfrak{C}(p,q)} S(p) \tag{1.2}$$

for $q \in \operatorname{Obj}\mathfrak{C}$, and for $f \in \mathfrak{C}(q,q')$ letting $(i_{\#}S)(f)\colon (i_{\#}S)(q) \to (i_{\#}S)(q')$ map the copy of $S(p)$ indexed by f identically to that indexed by gf. $i_{\#}\colon \mathfrak{Top}^{\operatorname{Obj}\mathfrak{C}} \to \mathfrak{Top}^{\mathfrak{C}}$ is made into a functor in the obvious way.

1.1.3 Definition. Let $X \in \mathfrak{Top}^{\mathfrak{C}}$. We call X a *free* $\mathfrak{C}$*-diagram*, if there exists a filtration $X = \bigcup_{n \geq 0} X_n$, $X_n \in \mathfrak{Top}^{\mathfrak{C}}$, such that X carries the final topology with respect to the inclusion maps $X_n \to X$, $X_0(q) = \emptyset$ for all $q \in \operatorname{Obj}\mathfrak{C}$, and for each $n \geq 1$ exist $A_n, B_n \in \mathfrak{Top}^{\operatorname{Obj}\mathfrak{C}}$, a map $j\colon B_n \to A_n$ consisting of closed cofibrations, and a diagram

$$\begin{array}{ccc} i_{\#}B_n & \longrightarrow & X_{n-1} \\ {\scriptstyle i_{\#}j}\big\downarrow & & \big\downarrow{\scriptstyle \text{incl.}} \\ i_{\#}A_n & \longrightarrow & X_n \end{array}$$

which is a pushout diagram.

1.1.4 Remark. What we call a free $\mathfrak{C}$-diagram is called a principal $\mathfrak{C}$-space in [FF89]. Indeed it can be argued that the term *free diagram* would better be reserved to those in the image of the functor $i_{\#}$. However, a chain complex is also called free, if it is free in every degree, and similarly free diagrams are made up of pieces in the image of $i_{\#}$. Also compare with Definition 1.2.2.

1.1.5 Definition. We define a $(\mathfrak{C} \times \mathfrak{C}^o)$-diagram of simplicial sets by

$$\begin{aligned} E\mathfrak{C}(q', q)_n &:= \left\{ (f_0, f_1, \ldots, f_{n+1})\colon f_i \in \mathfrak{C}, f_0 f_1 \cdots f_{n+1} \in \mathfrak{C}(q, q') \right\} \\ E\mathfrak{C}(g', g)_n \left((f_0, f_1, \ldots, f_{n+1}) \right) &:= (g' f_0, f_1, \ldots, f_{n+1} g) \end{aligned}$$

with boundaries

$$d^i(f_0, f_1, \ldots, f_{n+1}) := (f_0, \ldots, f_i f_{i+1}, \ldots, f_{n+1})$$

and degeneracies

$$s^i(f_0, f_1, \ldots, f_{n+1}) := (f_0, \ldots, f_i, \mathrm{id}, f_{i+1}, \ldots, f_{n+1}).$$

We define $E\mathfrak{C} \in \mathfrak{Top}^{\mathfrak{C} \times \mathfrak{C}^o}$ to be the simplicial realization of this diagram.

1.1.6 Remark. We think of a $X \in \mathfrak{Top}^{\mathfrak{C}\times\mathfrak{C}^o}$ as being equipped with commuting right and left operations of $\mathfrak{C}$. For example, for $Y \in \mathfrak{Top}^{\mathfrak{C}^o}$, $X \times_{\mathfrak{C}} Y$ is defined and inherits the left operation, i.e. $X \times_{\mathfrak{C}} Y \in \mathfrak{Top}^{\mathfrak{C}}$.

1.1.7 Proposition. *Let $X \in \mathfrak{Top}^{\mathfrak{C}}$. $E\mathfrak{C} \times_{\mathfrak{C}} X \in \mathfrak{Top}^{\mathfrak{C}}$ is a free $\mathfrak{C}$-diagram.*

Sketch of proof. The idea is to choose the pairs $(A_n(q), B_n(q))$ of Definition 1.1.3 as $(\Delta^n, \mathfrak{d}\Delta^n) \times \coprod_{q' \in \mathrm{Obj}\,\mathfrak{C}} (\{(f_1, \dots, f_n)\colon f_1 \cdots f_n \in \mathfrak{C}(q', q)\} \times X(q'))$. □

1.1.8 Proposition. *Let $X \in \mathfrak{Top}^{\mathfrak{C}}$. A point of $(E\mathfrak{C} \times_{\mathfrak{C}} X)(q)$ is determined by $f_0, \dots, f_{n+1}$ with $f_0 f_1 \cdots f_{n+1} \in \mathfrak{C}(q', q)$, $s \in \Delta^n$ and $x \in X(q')$. Sending this point to $X(f_0 f_1 \cdots f_{n+1})(x) \in X(q)$ yields a well-defined map of $\mathfrak{C}$-diagrams $E\mathfrak{C} \times_{\mathfrak{C}} X \to X$.*

1.1.9 Definition. For $X \in \mathfrak{Top}^{\mathfrak{C}}$ we define the *colimit of* X as (or identify it with) $\operatorname{colim} X := * \times_{\mathfrak{C}} X$ and we define the *homotopy colimit of* X as $\operatorname{hcolim} X := * \times_{\mathfrak{C}} E\mathfrak{C} \times_{\mathfrak{C}} X$, where $*$ denoted the constant $\mathfrak{C}^o$-diagram of spaces consisting of a single point.

The map from Proposition 1.1.8 induces a map $\operatorname{hcolim} X \to \operatorname{colim} X$.

1.1.10 Remark. For $X \in \mathfrak{Top}^{\mathfrak{C}^o}$ we have $\operatorname{hcolim} X = * \times_{\mathfrak{C}^o} E\mathfrak{C}^o \times_{\mathfrak{C}^o} X \approx X \times_{\mathfrak{C}} E\mathfrak{C} \times_{\mathfrak{C}} *$, and we will switch between both versions ad libitum.

1.1.11 Definition. Let $X, Y \in \mathfrak{Top}^{\mathfrak{C}}$, $f\colon X \to Y$ a map of $\mathfrak{C}$-diagrams. The map f is called a *homotopy equivalence*, if there exist $g\colon Y \to X$, $F\colon I \times X \to X$, $G\colon I \times Y \to Y$, all of them maps of $\mathfrak{C}$-diagrams, such that F is a homotopy from $g \circ f$ to id_X and G a homotopy from $f \circ g$ to id_Y.

1.1.12 Definition. Let $X, Y \in \mathfrak{Top}^{\mathfrak{C}}$, $f\colon X \to Y$ a map of $\mathfrak{C}$-diagrams. The map f is called a *weak homotopy equivalence*, if $f(q)\colon X(q) \to Y(q)$ is a homotopy equivalence for all $q \in \mathrm{Obj}\,\mathfrak{C}$.

1.1.13 Proposition. *The map $E\mathfrak{C} \times_{\mathfrak{C}} X \to X$ from Proposition 1.1.8 is a weak homotopy equivalence.*

See Lemma 1.2.9 for a proof of an algebraic analogue.

The following Proposition is our main technical tool. A proof can be found in [FF89, Thm 4.3] where it is attributed to [BV73].

1.1.14 Proposition. *Let $X, Y \in \mathfrak{Top}^{\mathfrak{C}}$, $f\colon X \to Y$ a weak homotopy equivalence. If X and Y are free diagrams, then f is a homotopy equivalence.*

1.1.15 Definition. Let $X, Y \in \mathfrak{Top}^{\mathfrak{C}^o}$. We will call a weak equivalence $f\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ a *topological ZŽ–map.*

1.1.16 Remark. A ZŽ–map $f\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ yields homotopy equivalences $X(q) \simeq (X \times_{\mathfrak{C}} E\mathfrak{C})(q) \simeq Y(q)$ and should be seen as a collection of homotopy equivalences $f_q\colon X(q) \xrightarrow{\simeq} Y(q)$ which not necessarily satisfy the equations $f_p \circ X(g) = Y(g) \circ f_q$ that would make these maps into a map of $\mathfrak{C}$-diagrams, but which satisfy these equations up to homotopies which fit together up to higher homotopies and so on. This will be made clearer for the algebraic analogue in Definition 1.2.12 and the calculations following it.

1.1.17 Proposition. *Let* $X, Y \in \mathfrak{Top}^{\mathfrak{C}^o}$ *and* $f\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ *be a ZŽ–map. If* Y *is a free* $\mathfrak{C}^o$*-diagram, then the induced map* $\operatorname{hcolim} X \approx X \times_{\mathfrak{C}} E\mathfrak{C} \times_{\mathfrak{C}} * \to Y \times_{\mathfrak{C}} * \approx \operatorname{colim} Y$ *is a homotopy equivalence.*

Proof. $X \times_{\mathfrak{C}} E\mathfrak{C}$ is a free diagram by Proposition 1.1.7, Y by assumption. Therefore f is a homotopy equivalence by Proposition 1.1.14. It follows that the induced map is a homotopy equivalence. □

1.1.18 Remark. Our usage of the term *ZŽ–map* is motivated by the fact that the homotopy equivalence to the link of a linear arrangement from its combinatorially defined homotopy model given by Ziegler and Živaljević [ZŽ93] can be seen to arise in this way. We present their homotopy model in Proposition 2.1.10.

Because of Proposition 1.1.13 we can note a special case.

1.1.19 Corollary. *Let* $X \in \mathfrak{Top}^{\mathfrak{C}}$*. If* X *is a free* $\mathfrak{C}$*-diagram, then the canonical map* $\operatorname{hcolim} X \to \operatorname{colim} X$ *is a homotopy equivalence.* □

We conlude this short overview with a simple proposition that allows us to compose ZŽ–maps.

1.1.20 Proposition. *Let* $f\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ *be a ZŽ–map. The map*

$$\begin{aligned} L(f)\colon X \times_{\mathfrak{C}} E\mathfrak{C} &\to Y \times_{\mathfrak{C}} E\mathfrak{C} \\ [(x,s)] &\mapsto [(f(x,s),s)] \end{aligned}$$

is a well defined weak homotopy equivalence (and hence homotopy equivalence) of $\mathfrak{C}^o$*-diagrams which makes the diagram*

$$\begin{array}{ccc} X \times_{\mathfrak{C}} E\mathfrak{C} & \xrightarrow{L(f)} & Y \times_{\mathfrak{C}} E\mathfrak{C} \\ & \searrow^{f} & \downarrow \\ & & Y \end{array} \tag{1.3}$$

commute. Consequently, if $g\colon Y \times_{\mathfrak{C}} E\mathfrak{C} \to Z$ *is another ZŽ–map, then so is* $g \circ L(f)\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Z$*.*

Proof. It is easily checked by computation that (1.3) is a well defined commutative diagram of $\mathfrak{C}^o$-diagrams. Since f and the natural map $Y \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ are weak homotopy equivalences, so is $L(f)$. By Proposition 1.1.7 and Proposition 1.1.14 $L(f)$ is even a homotopy equivalence. $g \circ L(f)$ is a composition of weak homotopy equivalences and therefore a weak homotopy equivalence. □

1.2 Diagrams of chain complexes

We introduce notation and terminology for diagrams of chain complexes similar to those introduced in the previous section for diagrams of spaces. In particular, free diagrams and algebraic ZŽ–maps will be defined. These tools will be used later on to study homology groups of arrangements, starting in Section 1.3.

The main results of this section will be proved by use of the spectral sequences of a certain double complex. It is one of these spectral sequences that will also be the basis for the proof of the graded formula for intersection products in an arrangement in a manifold, Proposition 1.3.20.

Homology

Let $\mathfrak{C}$ be a small category, R a hereditary ring.

1.2.1 Definition. Let $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $N \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$. We define $M \otimes_{\mathfrak{C}} N \in R\text{-}\mathfrak{Mod}$,

$$M \otimes_{\mathfrak{C}} N := \bigoplus_{q \in \operatorname{Obj} \mathfrak{C}} M(q) \otimes N(q) \Big/ K,$$

where K is the submodule generated by the elements $a \otimes N(f)(b) - M(f)(a) \otimes b$ for $a \in M(q)$, $b \in N(p)$, $f \in \mathfrak{C}(p, q)$.

1.2.2 Definition. For $S \in \mathfrak{Set}^{\operatorname{Obj} \mathfrak{C}}$, we define $F^{\mathfrak{C}} S \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ by

$$F^{\mathfrak{C}} S(q) := \mathrm{F}\left\{(f, s) \colon f \in \mathfrak{C}(p, q), s \in S(p)\right\},$$

and for $g \in \mathfrak{C}(q, q')$ letting $F^{\mathfrak{C}} S(g)$ be the morphism sending (f, s) to (gf, s). $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ is called a *free* $\mathfrak{C}$*-diagram of abelian groups* if M is isomorphic to $F^{\mathfrak{C}} S$ for some S, and $X \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ is called a *free* $\mathfrak{C}$*-diagram of chain complexes*, if X_n is a free diagram of abelian groups for every $n \in \mathbb{Z}$.

1.2.3 Lemma. *Let* $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $S \in \mathfrak{Set}^{\operatorname{Obj} \mathfrak{C}}$. *The map*

$$\bigoplus_{q \in \operatorname{Obj} \mathfrak{C}} \bigoplus_{s \in S(q)} M(q) \to M \otimes_{\mathfrak{C}} F^{\mathfrak{C}} S$$

which sends an element m *of the summand* $M(q)$ *indexed by* $s \in S(q)$ *to* $m \otimes (\mathrm{id}_q, s)$ *is an isomorphism.* □

1.2.4 Proposition. *If* $N \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ *is a free* $\mathfrak{C}$*–diagram, then the functor* $\bullet \otimes_{\mathfrak{C}} N \colon R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o} \to R\text{-}\mathfrak{Mod}$ *is exact.*

Proof. This follows from Lemma 1.2.3. □

1.2.5 Definition. We define $B(\mathfrak{C}) \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}\times\mathfrak{C}^o}$ by

$$B(\mathfrak{C})(q',q)_n := F\left\{q' \xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n+1}} q\right\},$$

$$B(\mathfrak{C})(g',g)(q' \xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n+1}} q) := p' \xleftarrow{g' f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n+1} g} p,$$

for $g \in \mathfrak{C}(p,q)$, $g' \in \mathfrak{C}(q',p')$,

$$\mathfrak{d}(q' \xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n+1}} q) := \sum_{k=0}^{n} (-1)^k\, q' \xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{k-1}} \cdot \xleftarrow{f_k f_{k+1}} \cdot \xleftarrow{f_{k+2}} \cdot \cdots \cdot \xleftarrow{f_{n+1}} q$$

1.2.6 Remark. $B(\mathfrak{C})(q',q)$ is the chain complex associated to the simplicial set $E\mathfrak{C}_\bullet$ of Definition 1.1.5.

1.2.7 Lemma. *$B(\mathfrak{C})$ is a free $\mathfrak{C}\times\mathfrak{C}^o$–diagram of chain complexes, $B(\mathfrak{C})_n \cong F^{\mathfrak{C}\times\mathfrak{C}^o} S_n$ with $S_n(q',q) := \left\{q' \xleftarrow{f_1} \cdots \xleftarrow{f_n} q\right\}$.* □

1.2.8 Definition and Proposition. *For $K \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ we define a map $\varepsilon \in \mathrm{Hom}_{\mathfrak{C}}(B(\mathfrak{C}) \otimes_{\mathfrak{C}} K, K)$ by*

$$\varepsilon\colon B(\mathfrak{C}) \otimes_{\mathfrak{C}} K \to K$$
$$\left(\xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{n+1}}\right) \otimes k \mapsto \begin{cases} 0, & n > 0, \\ K(f_0 f_1)(k), & n = 0. \end{cases}$$

This is a chain map if the K in second position is regarded as a chain complex concentrated in degree zero.

Proof. Since

$$\begin{aligned} \varepsilon\left(\mathfrak{d}\left(\left(\xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \xleftarrow{f_2}\right) \otimes k\right)\right) &= \varepsilon\left(\left(\xleftarrow{f_0 f_1} \cdot \xleftarrow{f_2}\right) \otimes k - \left(\xleftarrow{f_0} \cdot \xleftarrow{f_1 f_2}\right) \otimes k\right) \\ &= K(f_0 f_1 f_2)(x) - K(f_0 f_1 f_2)(x) = 0, \end{aligned}$$

ε is a chain map. □

1.2.9 Lemma. *The map ε from the preceding proposition is a chain homotopy equivalence. In other words, it is an acyclic resolution of K.*

Proof. We define

$$L\colon (B(\mathfrak{C})_r \otimes_{\mathfrak{C}} K)(q) \to (B(\mathfrak{C})_{r+1} \otimes_{\mathfrak{C}} K)(q)$$
$$\left(\xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{r+1}}\right) \otimes k \mapsto \left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{r+1}}\right) \otimes k$$

and calculate for $x = \left(\xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{r+1}}\right) \otimes k$, that

$$(\mathfrak{d}L + L\mathfrak{d})x = \begin{cases} x, & r > 0, \\ x - \left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{\mathrm{id}}\right) \otimes \varepsilon(x), & r = 0, \end{cases}$$

proving that $k \mapsto \left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{\mathrm{id}}\right) \otimes k$ is a homotopy inverse of ε. □

1.2.10 Proposition. *Let $X \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $K \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$, $X_p = 0$ for $p < 0$. If X is a free $\mathfrak{C}$-diagram, then the map*

$$H(X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K) \xrightarrow{H(\mathrm{id}_X \otimes \varepsilon)} H(X \otimes_{\mathfrak{C}} K)$$

is an isomorphism.

Proof. Again viewing the right hand side K as a complex concentrated in degree zero, the map $\mathrm{id}_X \otimes \varepsilon$ is a map of double complexes. Since X is free, the induced map $''H(X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K) \to {''H}(X \otimes_{\mathfrak{C}} K) = X \otimes_{\mathfrak{C}} K$, where $''H$ denotes homology with respect to the second differential of a double complex, is isomorphic to $\mathrm{id}_X \otimes H(\varepsilon)\colon X \otimes_{\mathfrak{C}} H(B(\mathfrak{C}) \otimes_{\mathfrak{C}} K) \to X \otimes_{\mathfrak{C}} H(K) = X \otimes_{\mathfrak{C}} K$ by Proposition 1.2.4, and the latter map is an isomorphism by Lemma 1.2.9. By [God58, Thm I.4.3.1] it follows that $H(\mathrm{id}_X \otimes \varepsilon)$ is also an isomorphism. □

1.2.11 Definition and Proposition. *We define the diagonal chain map $\Delta \in \mathrm{Hom}_{\mathfrak{C} \times \mathfrak{C}^o}(B(\mathfrak{C}), B(\mathfrak{C}) \otimes_{\mathfrak{C}} B(\mathfrak{C}))$ by*

$$\begin{aligned}\Delta(q' \xleftarrow{f_0} p_0 \xleftarrow{f_1} p_1 \cdots p_{n-1} \xleftarrow{f_n} p_n \xleftarrow{f_{n+1}} q) &:= \\ \sum_{k=0}^{n} q' \xleftarrow{f_0} p_0 \cdots p_{k-1} \xleftarrow{f_k} p_k \xleftarrow{id_{p_k}} p_k \otimes & \\ p_k \xleftarrow{id_{p_k}} p_k \xleftarrow{f_{k+1}} p_{k+1} \cdots p_n \xleftarrow{f_{n+1}} q.&\end{aligned}$$

Proof. This is a chain map by the usual calculation, additionally using

$$\begin{aligned}q' \xleftarrow{f_0} p_0 \cdots p_{k-1} \xleftarrow{f_k} p_k \otimes p_k \xleftarrow{id_{p_k}} p_k \xleftarrow{f_{k+1}} p_{k+1} \cdots p_n \xleftarrow{f_{n+1}} q = \\ = q' \xleftarrow{f_0} p_0 \cdots p_{k-2} \xleftarrow{f_{k-1}} p_{k-1} \xleftarrow{id_{p_{k-1}}} p_{k-1} \otimes p_{k-1} \xleftarrow{f_k} p_k \cdots p_n \xleftarrow{f_{n+1}} q\end{aligned}$$

here. □

1.2.12 Definition and Proposition. *Let $X, Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$. A chain map between diagrams $f \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y)$ induces chain maps*

$$\begin{aligned}f_q\colon X(q) &\to Y(q) \\ x &\mapsto f(x \otimes q \xleftarrow{\mathrm{id}_q} q \xleftarrow{\mathrm{id}_q} q).\end{aligned}$$

The maps $H(f_q)$ *form a homomorphism* $H(X) \xrightarrow{\cong} H(Y)$*, where* $H(X), H(Y) \in \left(R\text{-}\mathfrak{Mod}^{\mathbb{Z}}\right)^{\mathfrak{C}^o}$*. We call the map* f *an* algebraic ZŽ–map*, if for every* $q \in \mathfrak{C}$ *and every* R*-module* M *the map* $H(f_q)\colon H(X(q);M) \to H(Y(q);M)$ *is an isomorphism.*

Proof. Using the isomorphism

$$\eta\colon \operatorname{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y) \xrightarrow{\cong} \operatorname{Hom}_{\mathfrak{C}\times\mathfrak{C}^o}(B(\mathfrak{C}), \operatorname{Hom}(X,Y))$$

we can describe f_q by $f_q = \eta(f)\left(q \xleftarrow{\mathrm{id}_q} q \xleftarrow{\mathrm{id}_q} q\right)$. Now

$$\mathfrak{d} f_q = \eta(f)\left(\mathfrak{d}\left(q \xleftarrow{\mathrm{id}_q} q \xleftarrow{\mathrm{id}_q} q\right)\right) = 0,$$

i.e. f_q is a chain map. For $k \in \mathfrak{C}(p,q)$,

$$\begin{aligned}\mathfrak{d}\left(\eta(f)\left(\xleftarrow{\mathrm{id}_q} \cdot \xleftarrow{k} \cdot \xleftarrow{\mathrm{id}_p}\right)\right) &= \eta(f)\left(\xleftarrow{k} \cdot \xleftarrow{\mathrm{id}_p} - \xleftarrow{\mathrm{id}_q} \cdot \xleftarrow{k}\right) \\ &= f_p \circ X(k) - Y(k) \circ f_q,\end{aligned}$$

i.e. $f_p \circ X(k) \simeq Y(k) \circ f_q$. □

Looking at the preceding calculation from the other side, we get:

1.2.13 Proposition. *Let* $X, Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$*. Assume we are given chain maps* $f_q\colon X(q) \to Y(q)$ *for all* $q \in \mathfrak{C}$ *and for every* $k \in \mathfrak{C}(p,q)$ *a chain homotopy from* $Y(k) \circ f_q$ *to* $f_p \circ X(k)$*. If* $K_{kl} = Y(l) \circ K_k + K_l \circ X(k)$ *whenever* kl *is defined, then the map*

$$\begin{aligned} f\colon X \otimes_{\mathfrak{C}} B(\mathfrak{C}) &\to Y \\ x \otimes \left(\xleftarrow{k_1} p_0 \xleftarrow{k_0}\right) &\mapsto f_{p_0}(x \cdot k_1) \cdot k_0, \\ x \otimes \left(\xleftarrow{k_2} p_1 \xleftarrow{k_1} p_0 \xleftarrow{k_0}\right) &\mapsto (-1)^{|x|} K_{k_1}(x \cdot k_2) \cdot k_0, \\ x \otimes \left(\xleftarrow{k_0} \cdots \xleftarrow{k_{n+1}}\right) &\mapsto 0, \qquad n > 1 \end{aligned} \tag{1.4}$$

is a chain map of $\mathfrak{C}^o$*–diagrams as in Definition 1.2.12.*

Proof. f is a well-defined map of $\mathfrak{C}^o$–diagrams, and to check that it is a chain map, it will suffice to show that $\eta(f)$ is. We have

$$\begin{aligned}
\mathfrak{d}\left(\eta(f)\left(\xleftarrow{id_q} q \xleftarrow{id_q}\right)\right) &= \mathfrak{d} f_q = 0,\\
\mathfrak{d}\left(\eta(f)\left(\xleftarrow{id_q} q \xleftarrow{k} p \xleftarrow{id_p}\right)\right) &= \mathfrak{d} K_k = k \cdot k \cdot f_p - f_q \cdot k\\
&= \eta(f)\left(\xleftarrow{k} p \xleftarrow{\mathrm{id}_p} - \xleftarrow{id_q} q \xleftarrow{k}\right)\\
&= \eta(f)\left(\mathfrak{d}\left(\xleftarrow{id_q} q \xleftarrow{k} p \xleftarrow{id_p}\right)\right),\\
\eta(f)\left(\mathfrak{d}\left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{k} \cdot \xleftarrow{l} \cdot \xleftarrow{\mathrm{id}}\right)\right) &= \eta(f)\left(\xleftarrow{k} \cdot \xleftarrow{l} \cdot \xleftarrow{\mathrm{id}}\right) - \eta(f)\left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{kl} \cdot \xleftarrow{\mathrm{id}}\right)\\
&\quad + \eta(f)\left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{k} \cdot \xleftarrow{l}\right)\\
&= k \cdot K_l - K_{kl} + K_k \cdot l = 0,
\end{aligned}$$

proving this. □

We come to an algebraic analogue of Proposition 1.1.20.

1.2.14 Definition. Let $X, Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$ and $f \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y)$. We define $L(f) \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y \otimes_{\mathfrak{C}} B(\mathfrak{C}))$ as the composition

$$L(f)\colon X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \xrightarrow{\mathrm{id}\otimes\Delta} X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} B(\mathfrak{C}) \xrightarrow{f\otimes\mathrm{id}} Y \otimes_{\mathfrak{C}} B(\mathfrak{C}).$$

1.2.15 Proposition. *Let $X, Y, Z \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $f \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y)$, $g \in \mathrm{Hom}_{\mathfrak{C}^o}(Y \otimes_{\mathfrak{C}} B(\mathfrak{C}), Z)$. Then $h := g \circ L(f) \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Z)$, $L(h) = L(g) \circ L(f)$, and if f and g are ZŽ–maps, then so is h.*

Proof. $L(h) = L(g) \circ L(f)$ and $h_q = g_q \circ f_q$ are verified by calculation, using $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ for the first formula. □

1.2.16 Remark. Just in case that the equation $L(g) \circ L(f) = L(g \circ L(f))$ or the following diagram might seem oddly familiar to some readers, we remark that the functor $T\colon X \mapsto X \otimes_{\mathfrak{C}} B(\mathfrak{C})$ together with the natural transformations $\varepsilon'\colon T \to 1$ defined as in Definition 1.2.8 and $\mathrm{id} \otimes \Delta\colon T \to T^2$ is a comonad.

We now obtain an algebraic analogue of Proposition 1.1.17.

1.2.17 Proposition. *Let $X, Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $X_r = 0$, $Y_r = 0$ for $r < 0$, $K \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$, $f \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y)$. Then the diagram*

$$\begin{array}{ccc}
X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K & \xrightarrow{L(f)\otimes\mathrm{id}_K} & Y \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K\\
& \searrow^{f\otimes\mathrm{id}_K} & \big\downarrow{\scriptstyle \mathrm{id}_Y\otimes\varepsilon}\\
& & Y \otimes_{\mathfrak{C}} K
\end{array}$$

commutes. If f is a ZŽ–map, then $H(L(f) \otimes \mathrm{id}_K)$ is an isomorphism. Therefore, $H(f \otimes \mathrm{id}_K)$ is also an isomorphism, if additionally Y is a free diagram.

Proof. The commutativity of the diagram follows by calculation, the last sentence from Proposition 1.2.10. We have to check that $H(L(f)\otimes \mathrm{id}_K)$ is an isomorphism for a ZŽ–map f.

Let $x \in X(q)_r$, $c = \xleftarrow{\mathrm{id}_q} \cdot \xleftarrow{f_1} \cdots \xleftarrow{f_n} \cdot \xleftarrow{id_q} \in B(\mathfrak{C})(q,q')_s$, $k \in K(q')$. Then $(L(f)\otimes \mathrm{id}_K)(x\otimes c\otimes k) \in \bigoplus_{i=0}^{s} Y_{p+i}\otimes_{\mathfrak{C}} B(\mathfrak{C})_{s-i}\otimes_{\mathfrak{C}} K$. Since $L(f)\otimes\mathrm{id}_K$ respects the filtration by s, it induces a homomorphism between the corresponding spectral sequences. The term of $(L(f)\otimes \mathrm{id}_K)(x\otimes c\otimes k)$ in $X_r\otimes_{\mathfrak{C}} B(\mathfrak{C})_s\otimes_{\mathfrak{C}} K$ is $f_q(x)\otimes c\otimes k$. This describes the induced homomorphism between the E^1–terms $'H(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) \cong H(X\otimes K)\otimes_{\mathfrak{C}\times\mathfrak{C}^o} B(\mathfrak{C})$ and $'H(Y\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)\cong H(Y\otimes K)\otimes_{\mathfrak{C}\times\mathfrak{C}^o} B(\mathfrak{C})$ of the spectral sequences which is an isomorphism, because f is a ZŽ–map. By [God58, Thm I.4.3.1] it follows that $H(L(f)\otimes\mathrm{id}_K)$ is also an isomorphism. □

Proposition 1.2.17 will later be used to substitute for a diagram Y a simpler diagram X. We now examine conditions on X which make this especially worthwhile.

1.2.18 Proposition. *Let $X \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $K \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$, $K(q)$ a free R–module for all $q\in\mathfrak{C}$. If X is isomorphic to a direct sum $\bigoplus_{i\in I} X_i$, $X_i \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, such that there exist $n_i\in\mathbb{Z}$ with $H_r(X_i(q)) = 0$ for all $q\in\mathfrak{C}$, $r\neq n_i$, then $H(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)\cong H(H(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)$. This isomorphism is natural with respect to K and such that the diagram*

$$\begin{array}{ccc}
H_l(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})_0\otimes_{\mathfrak{C}} K & \longrightarrow & H_l(X\otimes_{\mathfrak{C}} B(\mathfrak{C})_0\otimes_{\mathfrak{C}} K) \\
\big\downarrow\!\!\!\!\downarrow & & \\
H_0(H_l(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) & & \big\downarrow \\
\big\downarrow & & \\
\bigoplus_r H_r(H_{l-r}(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) & \xrightarrow{\cong} & H_l(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)
\end{array} \tag{1.5}$$

commutes.

1.2.19 Remark. Most of the time we will actually be able to choose an X with zero differentials, in which case the proposition becomes trivial.

Proof. Because of the additivity of all of the involved constructions, we may assume that there exists an $n\in\mathbb{Z}$ such that $H_r(X) = 0$ for all $r\neq n$.

We will consider the spectral sequence of the double complex $X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K$ associated to the filtration by the degree of $B(\mathfrak{C})$. The E^1–term of this spectral sequence is $'H(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)$. Since $B(\mathfrak{C})\otimes_{\mathfrak{C}} K$ is a free $\mathfrak{C}$–diagram, it follows that $E^1_{r,s}\cong H_r(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})_s\otimes_{\mathfrak{C}} K$ and $E^2_{r,s}\cong H_s(H_r(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)$. Since $E^2_{r,s} = 0$ for $r\neq n$, $E^\infty = E^2$ and $H_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)\cong H_s(H_n(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)$, proving the existence of the isomorphism.

The isomorphism is given by the maps

$$\begin{aligned}
E^\infty_{n,s} \cong \frac{F_sH_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)}{F_{s-1}H_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)} &\xleftarrow{\cong} F_sH_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) \\
&\xrightarrow{\cong} H_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K),
\end{aligned}$$

where F denotes the induced filtration on the homology of the double complex. All of the maps are natural with respect to K.

The commutativity of the diagram is easily checked. □

1.2.20 Remark. If K does not consist of free modules, then the result will still hold, if we have that $H_r(X_i(q); K(q')) = 0$ for all $q, q' \in \mathfrak{C}$, $r \neq n_i$ and if we replace $H(H(X) \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K)$ by $H(H(X \otimes K) \otimes_{\mathfrak{C} \times \mathfrak{C}^o} B(\mathfrak{C}))$.

Products

When investigating intersection products in manifolds, many of the occurring chain complexes will carry products. We introduce some terminology to ease the description of these products.

We fix a functor $\lambda\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$.

1.2.21 Definition. Let $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$. A *$\lambda$-product* on M is a natural transformation from the functor

$$\begin{aligned} \mathfrak{C} \times \mathfrak{C} &\to R\text{-}\mathfrak{Mod} \\ (p, q) &\mapsto M(p) \otimes M(q) \end{aligned}$$

to the functor $M \circ \lambda$.

1.2.22 Definition and Proposition. *Let $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$ and $N \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ be equipped with λ-products, written by juxtaposition. Then*

$$\begin{aligned} (M \otimes_{\mathfrak{C}} N) \otimes (M \otimes_{\mathfrak{C}} N) &\to (M \otimes_{\mathfrak{C}} N) \\ (m \otimes n) \otimes (m' \otimes n') &\mapsto mm' \otimes nn' \end{aligned}$$

is well-defined, and we will usually equip $M \otimes_{\mathfrak{C}} N$ with this product. □

1.2.23 Definition and Proposition. *We define the cross product $\times$ on $B(\mathfrak{C})$, $\times \in \mathrm{Hom}_{(\mathfrak{C}\times\mathfrak{C})\times(\mathfrak{C}\times\mathfrak{C})^o}(B(\mathfrak{C}) \otimes B(\mathfrak{C}), B(\mathfrak{C} \times \mathfrak{C}))$, by*

$$\left(\xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdots\cdots \xleftarrow{f_k} \cdot \xleftarrow{f_{k+1}}\right) \otimes \left(\xleftarrow{g_0} \cdot \xleftarrow{g_1} \cdots\cdots \xleftarrow{g_l} \cdot \xleftarrow{g_{l+1}}\right) \mapsto$$

$$\sum_{\substack{(i_0,j_0)<\cdots<(i_{k+l},j_{k+l}) \\ (i_0,j_0)=(0,0) \\ (i_{k+l},j_{k+l})=(k,l) \\ i_{r+1}\le i_r+1 \\ j_{r+1}\le j_r+1}} \varepsilon^{i_0,\dots,i_{k+l}}_{j_0,\dots,j_{k+l}} \left(\xleftarrow{(f_0,g_0)} \cdot \xleftarrow{h^{i_0,i_1}_{j_0,j_1}} \cdots\cdots \xleftarrow{h^{i_{k+l-1},i_{k+l}}_{j_{k+l-1},j_{k+l}}} \cdot \xleftarrow{(f_{k+1},g_{k+1})} \right)$$

with $h^{i,i}_{j,j+1} = (\mathrm{id}, g_{j+1})$, $h^{i,i+1}_{j,j} = (f_{i+1}, \mathrm{id})$, $\varepsilon^{0,0,\dots,0}_{0,1,\dots,l} = \varepsilon^{0,1,\dots,k}_{0,0,\dots,0} = 1$, and the remaining $\varepsilon^{i_0,\dots,i_{k+l}}_{j_0,\dots,j_{k+l}} \in \{+1, -1\}$ determined by the requirement that $\times$ be a chain map.

The functor λ induces a natural transformation λ_* from $B(\mathfrak{C}\times\mathfrak{C})$ to $B(\mathfrak{C})\circ(\lambda\times\lambda^o)$ and the composition $x \otimes y \mapsto \lambda_*(x \times y)$ is a λ-product (more precisely a $(\lambda \times \lambda^o)$-product) on $B(\mathfrak{C})$ that we will sometimes denote by $\overset{\lambda}{\times}$. □

1.3 Arrangements

Let X be a topological space and $\mathcal{A}$ a set of subspaces of X. We set $Q := \{\bigcap S \colon S \subset \mathcal{A}\}$ and order Q by inclusion. The resulting partially ordered set Q will be considered a small category with a single arrow from p to q if $p \geq q$. We define $D \in \mathfrak{Top}^{Q^o}$ by $D(p) := p$ and letting $D(q \leftarrow p)$ be the inclusion from q to p.

1.3.1 Notation. The minimum map $\wedge \colon Q \times Q \to Q$, $p \wedge q = p \cap q$ is order preserving, hence a functor. Q has a minimum $\bigcap \mathcal{A}$ and a maximum $X = \bigcap \emptyset$. These will be denoted by $\bot$ and $\top$ respectively. For $p, q \in Q$, we denote by $[p, q]$ the *closed interval* $\{x \colon p \leq x \leq q\}$ and similar for open and half-open intervals. For a partially ordered set P, ΔP denotes the *order complex* of P, i.e. the simplicial complex with vertex set P and simplices all chains (totally ordered subsets) in P. By $C(\Delta P)$ we denote the ordered simplicial chain complex of the simplicial complex ΔP, for example $B(Q)(p, q) = C_*(\Delta[p, q])$.

Homotopy

When discussing homotopy properties of the arrangement $\mathcal{A}$ and the diagram D, we will assume that for all $q \in Q$ the inclusion map $\bigcup_{p<q} D(p) \to D(q)$ is a closed cofibration. The union on the left hand side can also be formulated as the colimit of the restrition of D to the poset $\{p \colon p < q\}$.

Under this hypothesis, we get the following proposition.

1.3.2 Proposition. *Let Q be finite. Then $D \in \mathfrak{Top}^{Q^o}$ is a free Q^o-diagram.*

Proof. We enumerate Q as $Q = \{q_1, \ldots, q_m\}$ with $i > j$ whenever $q_i > q_j$ and define diagrams $D^k \in \mathfrak{Top}^{Q^o}$ by

$$D^k(p) := \bigcup \{D(q_i) \colon i \leq k, q_i \leq p\} = \bigcup_{i \leq k} D(p \wedge q_i).$$

These D^k form a filtration of $D^m = D$, $D^0(p) = \emptyset$ for all $p \in Q$, and we will show that this filtration satisfies the conditions of Definition 1.1.3.

Let $0 < k \leq m$. For $p \geq q_k$, we have $D^k(p) = D^{k-1}(p) \cup D(q_k)$ and

$$D^{k-1}(p) \cap D(q_k) = \bigcup_{i<k} D(p \wedge q_i) \cap D(q_k) = \bigcup_{i<k} D(q_k \wedge q_i) = \bigcup_{q'<q_k} D(q').$$

The equation in the middle is the one that is special to a diagram derived from an arrangement and would not hold for an arbitrary diagram of inclusion maps. The key point is that the right hand side is independent of p. It follows that

$$\begin{array}{ccc} \bigcup_{q'<q_k} D(q') & \longrightarrow & D^{k-1}(p) \\ \downarrow & & \downarrow \\ D(q_k) & \longrightarrow & D^k(p) \end{array}$$

is a pushout diagram. For $p \not\geq q_k$, $D^k(p) = D^{k-1}(p)$ and therefore

$$\begin{array}{ccc} \emptyset & \longrightarrow & D^{k-1}(p) \\ \downarrow & & \downarrow \\ \emptyset & \longrightarrow & D^k(p) \end{array}$$

is a pushout diagram. Defining $A_k, B_k \in \mathfrak{Top}^{\operatorname{Obj} Q}$ by

$$A_k(p) := \begin{cases} D(q_k) & p = q_k, \\ \emptyset, & p \neq q_k, \end{cases} \qquad B_k(p) := \begin{cases} \bigcup_{q' < q_k} D(q') & p = q_k, \\ \emptyset, & p \neq q_k, \end{cases}$$

these combine to give a pushout diagram

$$\begin{array}{ccc} i_\# B_k & \longrightarrow & D^{k-1} \\ \downarrow & & \downarrow \\ i_\# A_k & \longrightarrow & D^k \end{array}$$

of Q^o-diagrams. □

1.3.3 Remark. From the construction in the proof it follows immediately that a diagram obtained from D by restriction to a sub-poset $\{q_1, \ldots, q_{m'}\}$ with $m' < m$ and q_i as in the proof is also free.

1.3.4 Remark. If X allows a triangulation such that every $A \in \mathcal{A}$ is a subcomplex, a condition that we will assume later when considering intersection products but that is less natural when considering homotopy theory, then a proof more along the lines of Proposition 1.3.9 is available to show that the diagram D is free. This would involve defining $A_k(p)$ to be the disjoint union of all k-simplices in $D(p)$ not contained in any $D(q)$ with $q < p$ and $B_k(p)$ the boundaries of those simplices. Finiteness of Q would not be needed.

Let us assume we are given the following data: A Q^o-diagram of spaces $E \in \mathfrak{Top}^{Q^o}$ (in applications this will be easier to describe than D and possibly carry less information), and for all $p \in Q$ maps

$$f^p \colon E(p) \times \Delta[p, \top] \to X$$

with $\operatorname{im} f^p|_{E(p)\times\Delta[p,q]} \subset D(q) = q$, $f^p(\cdot, \langle p \rangle) \colon E(p) \to D(p) = p$ a homotopy equivalence, and such that for $q \leq p$ the diagram

$$\begin{array}{ccccc} & & E(p) \times \Delta[p, \top] & & \\ & \overset{\mathrm{id}\times\mathrm{incl}}{\nearrow} & & \overset{f^p}{\searrow} & \\ E(p) \times \Delta[q, \top] & & & & X \\ & \underset{E(p \leftarrow q)\times\mathrm{id}}{\searrow} & & \underset{f^q}{\nearrow} & \\ & & E(q) \times \Delta[q, \top] & & \end{array} \tag{1.6}$$

commutes.

1.3.5 Proposition. *In the above situation and with Q finite, the maps f^p induce a homotopy equivalence* $\operatorname{hcolim} E' \xrightarrow{\simeq} \bigcup \mathcal{A}$*, where E' is the diagram obtained by restricting E to $Q \setminus \{\top\}$.*

Proof. We set $Q' := Q \setminus \{\top\}$ and denote the restriction of D to Q' by D'. D is free by Proposition 1.3.2, and so is D'. The inclusion $D'(q) \to \bigcup \mathcal{A}$ induce a map $\operatorname{colim} D' \to \bigcup \mathcal{A}$. Since $\bigcup \mathcal{A} = \bigcup_{q \in Q'} D'(q)$ and all of the $D'(q)$ are closed in $\bigcup \mathcal{A}$, this map is a homeomorphism.

Since in Q viewed as a category the composition of two morphisms is never an identity unless one of the original morphisms was (indeed both of them), degenerate simplices may be omitted in the construction of EQ, and $EQ(p,q) \approx \Delta[p,q]$. Hence the maps f^p are exactly what it takes to define a map of Q^o-diagrams $E \times_Q EQ \to D$, and they also define a map of $Q^{o\prime}$-diagrams $E' \times_Q' EQ' \to D'$. The assumption that $f^p(\cdot, \langle p \rangle)\colon E(p) \to p$ is a homotopy equivalence means that this map is a ZŽ–map. The diagram D' is free as shown in Proposition 1.3.2 respectively Remark 1.3.3. By Proposition 1.1.17 it follows that this ZŽ–map induces a homotopy equivalence $\operatorname{hcolim} E' \xrightarrow{\simeq} \operatorname{colim} D' \approx \bigcup \mathcal{A}$. □

1.3.6 Remark. For Proposition 1.3.5 it would not have been necessary to include $\bigcap \emptyset = X$ in Q when defining the maps f^p. Indeed, it may seem like a nuissance that we have included $\bigcap \emptyset = X$ in the definition of the intersection poset. However, when dealing with linear arrangements in Chapter 2 the constructed maps will naturally include the top element of the intersection poset and, more importantly, when turning to homology in the next section, we will also consider the relative case of $(X, \bigcup \mathcal{A})$ and for this the top element will be needed.

1.3.7 Example. Let us assume that every intersection of elements of $\mathcal{A}$ is either empty or contractible. As discussed in the preceding remark, we allow us to ignore the empty intersection. We set

$$E(q) := \begin{cases} \emptyset, & D(q) = \emptyset \\ *, & D(q) \neq \emptyset \end{cases}$$

and make E into a Q^o-diagram in the obvious and unique way. In this case $\operatorname{hcolim} E' \approx \Delta N$ where $N := \{q \in Q\colon D(q) \neq \emptyset,\ q < \top\}$. Maps f^p as above will automatically satisfy that $f^p(\cdot, \langle p \rangle)$ is a homotopy equivalence. Fulfilling the commutativity of (1.6) amounts to constructing a map $h\colon \Delta N \to \bigcup \mathcal{A}$ with $h[\langle q_0, \dots, q_r \rangle] \subset D(q_r)$ for all chains $q_0 < \cdots < q_r$ with $q_0 \in N$. Since $D(q)$ is contractible for every $q \in P$, such a map is easily defined by recursion over the skeleton of ΔP. By Proposition 1.3.5 the map h is a homotopy equivalence.

1.3.8 Remark. The fact that the diagram D is free can be seen as the reason for the appearance of the intersection poset in descriptions of the homotopy type of $\bigcup \mathcal{A}$ or the homology formulas in the next section.

An alternative construction of a diagram D for which Proposition 1.3.5 holds is to define the poset Q to be the power set of $\mathcal{A}$ ordered by reverse inclusion and to set $D(q) := \bigcap q$. This is again a free diagram, which is proven by repeating the proof of Proposition 1.3.2 verbatim. Vassiliev calls the diagram defined via the power set of $\mathcal{A}$ the *naive resolution* and the diagram defined via the intersection poset the *economical resolution* [Vas01].

When using the naive resolution, the space ΔN in Example 1.3.7 becomes the *nerve of* $\mathcal{A}$ and the result of the example the Nerve Theorem.

We will only consider the economical resultion in the following. Some of the applications to linear arrangements would work equally well for both kinds of resolutions, while for some the economical resolution is more practical. We will look at this again in Remark 2.1.26.

We will meet the naive resolution briefly once again in Section 2.4 in the guise of the atomic complex.

Homology

We will be interested in describing $H(\bigcup\mathcal{A})$ and $H(X,\bigcup\mathcal{A})$. For simplicity, we assume $\bigcup\mathcal{A}\neq X$. We assume the inclusion $\sum_{A\in\mathcal{A}} S(A)\to S(\bigcup\mathcal{A})$, where S denotes the singular chain complex, to induce an isomorphism in homology. Again we will write $S(D)$ instead of $S\circ D$ for the diagram of chain complexes arising from D by applying the singular chain functor.

1.3.9 Proposition. *$S(D)\in \mathfrak{d}R\text{-}\mathfrak{Mod}^{Q^o}$ is a free Q^o–diagram.*

Proof. Let $p\in Q$ and $\sigma\colon \Delta^r\to D(p)$ be a singular simplex. We set $q_\sigma := \bigcap\{q'\in Q\colon \operatorname{im}\sigma\subset q'\}$. Then $q_\sigma\leq p$ and $\sigma\in S_r(q_\sigma)$. It follows that $S_r(D)$ is freely generated by the system $(\{\sigma\colon\Delta^r\to D(p)\colon q_\sigma=p\})_{p\in Q}$. □

1.3.10 Definition. We define $K^u, K^p\in R\text{-}\mathfrak{Mod}^Q$ by

$$K^u(q) := \begin{cases}0, & q=\top,\\ R, & q<\top,\end{cases} \qquad K^u(q'\to q) := \begin{cases}0, & q'=\top,\\ \mathrm{id}_R, & q'<\top,\end{cases}$$

$$K^p(q) := \begin{cases}R, & q=\top,\\ 0, & q<\top,\end{cases} \qquad K^p(q'\to q) := \begin{cases}\mathrm{id}_R, & q'=q=\top,\\ 0, & q<\top.\end{cases}$$

The notation is chosen because of the following connection of K^u and K^p with the singular chain complexes of the union $\bigcup\mathcal{A}$ and the pair $(X,\bigcup\mathcal{A})$, respectively.

1.3.11 Proposition. *The chain maps*

$$S(D)\otimes_Q K^u\to S\left(\bigcup\mathcal{A}\right), \qquad S(D)\otimes_Q K^p\to S\left(X,\bigcup\mathcal{A}\right)$$
$$c\otimes k\mapsto kc \qquad\qquad c\otimes k\mapsto kc$$

are well defined and they induce isomorphisms $H(S(D)\otimes_Q K^u)\cong H(\bigcup\mathcal{A})$ and $H(S(D)\otimes_Q K^p)\cong H(X,\bigcup\mathcal{A})$.

Proof. By the proof of Proposition 1.3.9 and Lemma 1.2.3,

$$S_r(D) \otimes_Q K^u \cong \bigoplus_{p<\top} \bigoplus_{\substack{\sigma \in S_r(X) \\ q_\sigma = p}} R = \bigoplus_{\substack{\sigma \in S_r(X) \\ q_\sigma \in Q\setminus\{\top\}}} R = \sum_{A\in\mathcal{A}} S(A) =: S\mathcal{A},$$

and the first induced map factorizes as $H(S_r(D) \otimes_Q K^u) \xrightarrow{\cong} H(S\mathcal{A}) \xrightarrow{\cong} H(\bigcup \mathcal{A})$. Similarly, $S_r(D) \otimes_Q K^p$ is free on the r–simplices σ with $q_\sigma = \top$, and the second induced map factorizes as $H(S_r(D) \otimes_Q K^p) \xrightarrow{\cong} H(SX/S\mathcal{A}) \xrightarrow{\cong} H(X, \bigcup \mathcal{A})$. □

1.3.12 Proposition. *In the situation described in* (1.6)*, with the condition on* $f^p(\cdot, \langle p\rangle)\colon E(p) \to p$ *weakened to induce an isomorphism in homology,*

$$\begin{gathered} g\colon S(E) \otimes_Q B(Q) \to S(D) \\ c \otimes p \leftarrow q_0 \leftarrow \cdots \leftarrow q_n \leftarrow p' \mapsto f^p_*(c \times \langle q_0, \ldots, q_n\rangle), \end{gathered}$$

defines a ZŽ–map. For $K \in R\text{-}\mathfrak{Mod}^Q$ *the map* $g_*\colon H(S(E) \otimes_Q B(Q) \otimes_Q K) \to H(S(D) \otimes_Q K)$ *is an isomorphism.*

Proof. Because $\operatorname{im} f^p|_{E(p)\times\Delta[p,p']} \subset p'$, $f^p_*(c \times \langle q_0, \ldots, q_n\rangle)$ is in $S(D)(p) = S(p)$. It is well-defined because of the commutativity of (1.6). The map is a map of Q^o–diagrams, because the right hand side is independent of p'. That it is a chain map is now easily checked.

The map $S(E)(p) \to S(D)(p) = S(p)$, $c \mapsto g(c \otimes p \leftarrow p \leftarrow p) = f^p_*(c \times \langle p\rangle)$ induces an isomorphism in homology by assumption, so g is a ZŽ–map. The map g_* is an isomorphism by Proposition 1.2.17 and Proposition 1.3.9. □

In this situation, one may be lucky and able to prove that $H(S(E) \otimes_Q B(Q) \otimes_Q K)$ is isomorphic to $H(H(E) \otimes_Q B(Q) \otimes_Q K)$, e.g. by Proposition 1.2.18. It then follows that $H(S(D) \otimes_Q K) \cong H(H(D) \otimes_Q B(Q) \otimes_Q K)$.

The preceding results, and those in the section to come, are easily extended to the relative case. We will formulate and prove only the key step.

1.3.13 Proposition. *Let* $Y \subset X$ *and assume that the inclusion maps*

$$\begin{gathered} S(Y) + S\left(\bigcup \mathcal{A}\right) \to S\left(Y \cup \bigcup \mathcal{A}\right), \\ \sum_{A\in\mathcal{A}} S(A \cap Y) \to S\left(Y \cap \bigcup \mathcal{A}\right) \end{gathered}$$

also induce isomorphisms in homology. Let D' *be the* Q^o*–diagram of pairs of spaces defined by* $D'(q) := (q, q \cap Y)$*. Then* $S(D')$ *is a free* Q^o*–diagram, and the chain maps*

$$\begin{array}{cc} S(D') \otimes_Q K^u \to S\left(\bigcup \mathcal{A}, Y \cap \bigcup \mathcal{A}\right), & S(D') \otimes_Q K^p \to S\left(X, Y \cup \bigcup \mathcal{A}\right) \\ c \otimes k \mapsto kc & c \otimes k \mapsto kc \end{array}$$

are well defined and induce isomorphisms $H\left(\bigcup\mathcal{A}, Y\cap\bigcup\mathcal{A}\right)\cong H(S(D')\otimes_Q K^u)$ *and* $H\left(X, Y\cup\bigcup\mathcal{A}\right)\cong H(S(D')\otimes_Q K^p)$.

Let $\bar{Q} := \{p\in Q\colon D(p)\not\subset Y\}$ *and* $\bar{D}'$ *the* $\bar{Q}^o$*–diagram obtained by restricting* D'. *Then* $S(\bar{D}')$ *is a free diagram and for any* $K\in\mathfrak{Ab}^Q$ *the obvious map* $S(\bar{D}')\otimes_{\bar{Q}}\bar{K}\to S(D')\otimes_Q K$, *where* $\bar{K}$ *is the restriction of* K, *is an isomorphism. In particular,* $H\left(\bigcup\mathcal{A}, Y\cap\bigcup\mathcal{A}\right)\cong H(S(\bar{D}')\otimes_{\bar{Q}} K^u)$ *and* $H\left(X, Y\cup\bigcup\mathcal{A}\right)\cong H(S(\bar{D}')\otimes_{\bar{Q}} K^p)$.

Proof. Taking up the notation of the proof of Proposition 1.3.9, $S_r(D')$ is free on the system $\left(\{\sigma\colon\Delta^r\to D(p)\colon q_\sigma = p, \operatorname{im}\sigma\not\subset Y\}\right)_{p\in Q}$, and as in Proposition 1.3.11 the first induced map factorizes as $H(S(D')\otimes_Q K^u)\xrightarrow{\cong} H(S\mathcal{A}/(S\mathcal{A}\cap SY))\xrightarrow{\cong} H\left(\bigcup\mathcal{A}, Y\cap\bigcup\mathcal{A}\right)$. $S(D')\otimes_Q K^p$ is free on the singular simplices neither in Y nor in any of the $A\in\mathcal{A}$, and the second induced map factorizes as $H(S(D')\otimes_Q K^p)\xrightarrow{\cong} H(SX/(SY\cup S\mathcal{A}))\xrightarrow{\cong} H\left(X, Y\cup\bigcup\mathcal{A}\right)$.

To justify the claims regarding $\bar{Q}$, it suffices to remark that the free diagram $S_r(D')$ has no generators for $p\in Q\setminus\bar{Q}$. □

Intersection products in manifolds

If X is a compact n-dimensional manifold oriented over R, we are interested in the intersection products $\bullet$ defined by commutativity of

$$\begin{array}{ccc} H_k(X,\bigcup\mathcal{A})\otimes H_l(X,\bigcup\mathcal{A}) & \xrightarrow{\quad\bullet\quad} & H_{k+l-n}(X,\bigcup\mathcal{A}) \\ {\scriptstyle\frown[X]}\Big\uparrow{\scriptstyle\cong} & & {\scriptstyle\frown[X]}\Big\uparrow{\scriptstyle\cong} \\ H^{n-k}(X\setminus\bigcup\mathcal{A})\otimes H^{n-l}(X\setminus\bigcup\mathcal{A}) & \xrightarrow{\ \smile\ } & H^{2n-k-l}(X\setminus\bigcup\mathcal{A}) \end{array}$$

and

$$\begin{array}{ccc} H_k(\bigcup\mathcal{A})\otimes H_l(\bigcup\mathcal{A}) & \xrightarrow{\quad\bullet\quad} & H_{k+l-n}(\bigcup\mathcal{A}) \\ {\scriptstyle\frown[X]}\Big\uparrow{\scriptstyle\cong} & & {\scriptstyle\frown[X]}\Big\uparrow{\scriptstyle\cong} \\ H^{n-k}(X,X\setminus\bigcup\mathcal{A})\otimes H^{n-l}(X,X\setminus\bigcup\mathcal{A}) & \xrightarrow{\ \smile\ } & H^{2n-k-l}(X,X\setminus\bigcup\mathcal{A}). \end{array}$$

For Poincaré duality to hold and for technical reasons, we assume X to allow a triangulation such that all $A\in\mathcal{A}$ are subcomplexes.

In this section we will see what information about the intersection products can obtained algebraically without special geometric knowledge of the class of arrangements at hand. For linear arrangements, this will yield the graded formulas of Section 2.2.

For the description of these products it will be important that there is a product on $C_*(\Delta Q)$.

1.3.14 Definition and Remark. *Let* $c \in C_r(\Delta Q)$*,* $d \in C_s(\Delta Q)$*. Then we have* $c \times d \in C_{r+s}(\Delta Q \times \Delta Q) = C_{r+s}(\Delta(Q \times Q))$ *and* $\wedge_*(c \times d) \in C_{r+s}(Q)$*, since* $\wedge \colon Q \times Q \to Q$ *is order preserving and hence a simplicial map* $\Delta(Q \times Q) \to \Delta Q$*. If* $c = \langle p_0, \dots, p_r \rangle$ *and* $d = \langle q_0, \dots, q_s \rangle$*, then* $\wedge_*(c \times d)$ *is a linear combination of simplices with first vertex* $p_0 \wedge q_0$ *and last vertex* $p_r \wedge q_s$*. This specializes Definition 1.2.23 with* $\wedge$ *for* λ*, and as there we will set* $c \mathbin{\hat{\times}} d := \wedge_*(c \times d)$*. The multiplication in* R *defines* $\wedge$*-products (see Definition 1.2.21) on* K^u *and* K^p *in the obvious way. These products and Definition 1.2.22 will be used to define products on several chain complexes.*

1.3.15 Proposition. *Let* K *be equipped with a* $\wedge$*-product in the above situation. The spectral sequence of the filtration of* $S(D) \otimes_Q B(Q) \otimes_Q K$ *by the grading of* $B(Q)$ *can be made into a multiplicative* E^1*-spectral sequence with the multiplication on* E^1 *isomorphic to the multiplication on* $H(D) \otimes_Q B(Q) \otimes_Q K$ *given by*

$$\begin{aligned}(a \otimes \langle p_0, \dots, p_r \rangle \otimes m) &\otimes (b \otimes \langle q_0, \dots, q_s \rangle \otimes m') \\ &\mapsto (-1)^{r(n-l)}[(a \bullet b) \otimes (\langle p_0, \dots, p_r \rangle \mathbin{\hat{\times}} \langle q_0, \dots, q_s \rangle) \otimes (m \cdot m'), \qquad (1.7)\end{aligned}$$

where $a \in H_k(D(p))$*,* $b \in H_l(D(q))$*,* $a \bullet b \in H_{k+l-n}(D(p \wedge q))$*,* $m \in K(p')$*,* $m' \in K(q')$*,* $m \cdot m' \in K(p' \wedge q')$*. The multiplication is a chain map of degree* $(-n, 0)$*. If* $K = K^u$ *or* $K = K^p$ *the multiplication on* E^∞ *is induced, via the isomorphism* $H(S(D) \otimes_Q B(Q) \otimes_Q K^u) \cong H(\bigcup \mathcal{A})$ *or* $H(S(D) \otimes_Q B(Q) \otimes_Q K^p) \cong H(X, \bigcup \mathcal{A})$ *respectively, by the intersection product on* $H(\bigcup \mathcal{A})$ *or* $H(X, \bigcup \mathcal{A})$*.*

1.3.16 Remark. The intersection product $a \bullet b$ in the above proposition is defined by commutativity of

$$\begin{array}{ccc} H_k(A) \otimes H_l(B) & \xrightarrow{\quad\bullet\quad} & H_{k+l-n}(A \cap B) \\ {\scriptstyle \frown[X]}\uparrow{\scriptstyle \cong} & & {\scriptstyle \frown[X]}\uparrow{\scriptstyle \cong} \\ H^{n-k}(X, X \setminus A) \otimes H^{n-l}(X, X \setminus A) & \xrightarrow{\smile} & H^{2n-k-l}(X, (X \setminus A) \cup (X \setminus B)). \end{array}$$

Proof of Proposition 1.3.15

We will from now on consider X to be triangulated by a barycentric subdivision of a triangulation of which all $A \in \mathcal{A}$ are subcomplexes. This will make all $p \in Q$ full subcomplexes of X. We will denote the face poset of this triangulation by FX and by $C(FX)$ the chain complex of ascending (from 0-simplices to n-simplices) chains in FX.

1.3.17 Definition and Proposition. *For a subcomplex* A *of* X*, cap products*

$$\begin{aligned} C^r(FX, FX \setminus FA) \otimes C_s(FX) &\xrightarrow{\frown} C_{s-r}(FA) \\ C^r(FX \setminus FA) \otimes C_s(FX) &\xrightarrow{\frown} C_{s-r}(FX, FA) \\ h \otimes \langle f_0, \dots, f_s \rangle &\mapsto (-1)^{r(s-r)} h(\langle f_{s-r}, \dots, f_s \rangle) \langle f_0, \dots, f_{s-r} \rangle \end{aligned}$$

are defined, where

$$C^r(FX, FX \setminus FA) = \ker\left(\mathrm{Hom}(C_r(FX), R) \to \mathrm{Hom}(C_r(FX \setminus FA), R)\right),$$
$$C_{s-r}(FX, FA) = \mathrm{coker}\left(C_{s-r}(FA) \to C_{s-r}(FX)\right).$$

Proof. If $\langle f_0, \ldots, f_{s-r}\rangle$ is not in $C_*(FA)$, then f_{s-r} is not in A and therefore $\langle f_{s-r}, \ldots, f_s\rangle$ is in $C_*(FX \setminus FA)$, so that $h(\langle f_{s-r}, \ldots, f_s\rangle) = 0$ for the first kind of product or $h(\langle f_{s-r}, \ldots, f_s\rangle)$ is defined for the second kind of product. $\square$

Since $\Delta(FX)$ is just the barycentric subdivison of X, we have $H(C(FX)) \cong H(X)$. Let $o \in C_n(FX)$ represent the orientation class $[X] \in H_n(X)$. Regarding $C(FA)$ as a subcomplex of the singular chain complex $S(A)$, this yields a map $C(FX, FX \setminus FA) \xrightarrow{\frown o} S(A)$ which induces an isomorphism in homology, if A is a full subcomplex. $\Delta(FX \setminus FA)$ is the subcomplex of the barycentric subdivision of X that consists of all simplices which do not meet A. It is the complement of an open normal neighbourhood of A.

1.3.18 Definition and Proposition. *If A, B are subcomplexes of X, a cup product*

$$C^r(FX, FX \setminus FA) \otimes C^s(FX, FX \setminus FB) \to C^{r+s}(FX, FX \setminus F(A \cap B))$$
$$g \otimes h \mapsto g \smile h,$$

$(g \smile h)(\langle f_0, \ldots, f_{r+s}\rangle) := (-1)^{rs} g(\langle f_0, \ldots, f_r\rangle) h(\langle f_r, \ldots, f_{r+s}\rangle)$, *is defined.*

Proof. If $\langle f_0, \ldots, f_{r+s}\rangle$ is in $C_*(FX \setminus F(A \cap B))$, then $f_0 \notin F(A \cap B)$ and therefore either $f_0 \notin FA$ and $\langle f_0, \ldots, f_r\rangle \in C(FX \setminus FA)$ or $f_0 \notin FB$ and $\langle f_r, \ldots, f_{r+s}\rangle \in C(FX \setminus FB)$. In either case $(g \smile h)(\langle f_0, \ldots, f_{r+s}\rangle) = 0$. $\square$

We now define $Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{Q^o}$ by $Y(p)_r := C^{-r}(FX, FX \setminus Fp)$. We equip this with the $\wedge$–product given by the cup product just defined. This also defines products on $Y \otimes_Q K$ and $Y \otimes_Q B(Q) \otimes_Q K$ by Definition 1.2.22 and Definition 1.2.23.

The product on the double complex $Y \otimes_Q B(Q) \otimes_Q K$ makes its second spectral sequence into a multiplicative spectral sequence with the multiplication on the E^1-term $H(Y) \otimes_Q B(Q) \otimes_Q K$ isomorphic to

$$[\alpha \otimes \langle p_0, \ldots, p_r\rangle \otimes m] \otimes [\beta \otimes \langle q_0, \ldots, q_s\rangle \otimes m']$$
$$\mapsto (-1)^{rl}[(\alpha \smile \beta) \otimes (\langle p_0, \ldots, p_r\rangle \hat{\times} \langle q_0, \ldots, q_s\rangle) \otimes (m \cdot m'),$$

where $\alpha \in H^k(X, X \setminus p)$, $\beta \in H^l(X, X \setminus q)$, $\alpha \smile \beta \in H^{k+l}(X, X \setminus (p \cap q))$, $m \in K(p')$, $m' \in K(q')$, $m \cdot m' \in K(p' \wedge q')$.

Now $\frown o$ is a Q^o-chain-map (of degree n) from Y to $S(D)$, inducing isomorphisms

$$H^*(X, X \setminus p) \cong H(Y(p)) \xrightarrow[\cong]{\frown [X]} H(S(D(p))) = H_*(p)$$

for all p. It therefore induces an isomorphism between the second spectral sequences of the double complexes $Y \otimes_Q B(Q) \otimes_Q K$ and $S(D) \otimes_Q B(Q) \otimes_Q K$ from the E^1-terms on. We use this isomorphism to make the spectral sequence of $S(D) \otimes_Q B(Q) \otimes_Q K$ multiplicative. Since $\frown [X]$ takes cup products into intersection products, this already proves the first part of the proposition.

1.3.19 Proposition. *The maps*

$$\begin{aligned} Y \otimes_Q K^u &\to C^*(FX, FX \setminus F \bigcup \mathcal{A}), \\ Y \otimes_Q K^p &\to C^*(FX \setminus F \bigcup \mathcal{A}), \\ [f \otimes k] &\mapsto kf \end{aligned}$$

are well defined and respect products.

Proof. $Y(\top) \otimes K^u(\top) = 0$ and for $q < \top$ the complex $Y(q) = C^*(FX, FX \setminus Fq)$ is a subcomplex of $C^*(FX, FX \setminus F \bigcup \mathcal{A})$, since $q \subset \bigcup \mathcal{A}$. Therefore the map $Y \otimes_Q K^u \to C^*(FX, FX \setminus F \bigcup \mathcal{A})$ is well defined.

$Y(q) \otimes K^p(q) = 0$ for $q < \top$ and $Y(\top) = C^*(FX)$ restricts to $C^*(FX \setminus F \bigcup \mathcal{A})$. Let $q < \top$, $k \in K^p(\top)$, $f \in Y(q) = C^*(FX, FX \setminus Fq)$. Then $Y(q \leftarrow \top) f \otimes k = f \otimes K^p(q \leftarrow \top) k = 0$ and f restricts to 0 in $C^*(FX \setminus F \bigcup \mathcal{A})$. Therefore the map $Y \otimes_Q K^p \to C^*(FX \setminus F \bigcup \mathcal{A})$ is well defined.

Both maps respect products because of the naturality of cup products. □

The multiplication on E^∞ is induced by the multiplication on the homology of the double complex. In the commutative diagram

$$\begin{array}{ccc} H(Y \otimes_Q B(Q) \otimes_Q K^p) & \xrightarrow[\cong]{H((\frown o)\otimes \mathrm{id} \otimes \mathrm{id})} & H(S(D) \otimes_Q B(Q) \otimes_Q K^p) \\ \big\downarrow {\scriptstyle H(\mathrm{id}_Y \otimes \varepsilon)} & & {\scriptstyle \cong} \big\downarrow {\scriptstyle H(\mathrm{id}_{S(D)} \otimes \varepsilon)} \\ H(Y \otimes_Q K^p) & \xrightarrow[\cong]{H((\frown o)\otimes \mathrm{id})} & H(S(D) \otimes_Q K^p) \\ \big\downarrow & & {\scriptstyle \cong} \big\downarrow \\ H^*(X \setminus \bigcup \mathcal{A}) & \xrightarrow{\frown [X]} & H_*(X, \bigcup \mathcal{A}) \end{array}$$

the maps on the left are ring homomorphisms, while the map at the bottom takes cup products into cap products. This proves the second part of the proposition for K^p and $H_*(X, \bigcup \mathcal{A})$. The corresponding diagram with K^u, $H^*(X, X \setminus \bigcup \mathcal{A})$, and $H_*(\bigcup \mathcal{A})$ completes the proof of Proposition 1.3.15 □

Product formulas

We apply Proposition 1.3.15 to a class of arrangements for which the E^2-term of the spectral sequence is isomorphic to the homology of the arrangement.

1.3.20 Proposition. *Assume that there is* $Z \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{Q^o}$ *satisfying the condition of Proposition 1.2.18 and a* $Z\check{Z}$*–map* $\zeta\colon Z \otimes_Q B(Q) \to S(D)$*. Then*

$$\begin{aligned} H(H(D) \otimes_Q B(Q) \otimes_Q K) &\overset{\cong}{\longleftarrow} H(H(Z) \otimes_Q B(Q) \otimes_Q K) \\ &\underset{\alpha}{\overset{\cong}{\longrightarrow}} H(Z \otimes_Q B(Q) \otimes_Q K) \qquad (1.8) \\ &\overset{\cong}{\longrightarrow} H(S(D) \otimes_Q B(Q) \otimes_Q K), \end{aligned}$$

the isomorphism α *being that from Proposition 1.2.18 (and trivial, if the boundary map in* Z *equals zero) and the other two induced by* ζ*: The first arrow by the isomorphism* $H(Z) \cong H(D)$ *described in Definition 1.2.12 and the last one by Proposition 1.2.17. We denote the composition of these isomorphisms by* $\tilde{\phi}$ *and decompose the resulting isomorphism* $\phi\colon H(H(D) \otimes_Q \otimes B(Q) \otimes_Q K^p) \to H_*(X, \bigcup \mathcal{A})$ *as* $\phi = \sum_k \phi_k$ *with*

$$\phi_k\colon H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^p) \to H_{k+r}\left(X, \bigcup \mathcal{A}\right). \qquad (1.9)$$

Then

$$\phi_k(x) \bullet \phi_l(y) - \phi_{k+l-n}(x \cdot y) \in \bigoplus_{i>k+l-n} \operatorname{im} \phi_i, \qquad (1.10)$$

where $x \cdot y$ *denotes the product induced by* (1.7)*.*

Proof. The filtration defining the spectral sequence in Proposition 1.3.15 induces a filtration on $H(S(D) \otimes_Q B(Q) \otimes_Q K^p)$ and we will first identify its image in $H_*(X, \bigcup \mathcal{A})$. On $H(Z \otimes_Q B(Q) \otimes_Q K^p)$ the filtration given by the degree of $B(Q)$ is

$$F_k(H_t(Z \otimes_Q B(Q) \otimes_Q K^p)) = \bigoplus_{i=0}^{k} \alpha\left[H_i(H_{t-i}(Z) \otimes_Q B(Q) \otimes_Q K^p)\right].$$

The map $L(\zeta) \otimes \mathrm{id}_{K^p}$ induces an isomorphism respecting filtrations, therefore the filtration on $H_*(X, \bigcup \mathcal{A})$ induced by that on $H(S(D) \otimes_Q B(Q) \otimes_Q K^p)$ is

$$F_k\left(H_t\left(X, \bigcup \mathcal{A}\right)\right) = \bigoplus_{i=0}^{k} \phi_{t-i}\left[H_i(H_{t-i}(D) \otimes_Q B(Q) \otimes_Q K^p)\right].$$

For $x \in H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^p)$, $\tilde{\phi}(x) \in F_r(H_{k+r}(S(D) \otimes_Q B(Q) \otimes_Q K^p))$. The E^∞-term $F_r(H_{k+r}(S(D) \otimes_Q B(Q) \otimes_Q K^p))/F_{r-1}(H_{k+r}(S(D) \otimes_Q B(Q) \otimes_Q K^p))$ equals the E^2-term $H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^p)$, and the class that $\tilde{\phi}(x)$ represents in the E^∞-term is again x. From this and Proposition 1.3.15 it follows that for $y \in H_s(H_l(D) \otimes_Q B(Q) \otimes_Q K^p)$ we have

$$\begin{aligned} \phi_k(x) \bullet \phi_l(y) - \phi_{k+l-n}(x \cdot y) &\in F_{r+s-1}\left(H_{k+l-n+r+s}\left(X, \bigcup \mathcal{A}\right)\right) \\ &\subset \bigoplus_{i=0}^{r+s-1} \operatorname{im} \phi_{k+l-n+r+s-i} = \bigoplus_{i=1}^{r+s} \operatorname{im} \phi_{k+l-n+i} \subset \bigoplus_{i>k+l-n} \operatorname{im} \phi_i \end{aligned}$$

as stated. □

In Proposition 1.3.15 the case of $K = K^u$, i.e. of intersection products in $H_*(\bigcup \mathcal{A})$, is the less interesting one.

1.3.21 Proposition. *The product*

$$H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^u) \otimes H_s(H_l(D) \otimes_Q B(Q) \otimes_Q K^u) \\ \to H_{r+s}(H_{k+l-n}(D) \otimes_Q B(Q) \otimes_Q K^u)$$

is zero except for $r = s = 0$.

Proof. Let $x \in H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^u)$, $y \in H_s(H_l(D) \otimes_Q B(Q) \otimes_Q K^u)$. The sequence

$$H_{r+1}(H_k(D) \otimes_Q B(Q) \otimes_Q R^p) \xrightarrow{\partial} H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^u) \\ \to H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K)$$

is exact. If $r > 0$, then $H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K) = 0$ and therefore $z \in H_{r+1}(H_k(D) \otimes_Q B(Q) \otimes_Q R^p)$ exists with $\partial z = x$. It is easily checked that with the obvious definition of the $\wedge$–product $R^p \otimes R^u \to R^u$ it follows that $xy = (\partial z)y = \partial(zy) = 0$. □

We will, however, often be able to get a better result on intersection products in $H_*(\bigcup \mathcal{A})$ without resorting to Proposition 1.3.15.

1.3.22 Proposition. *We assume the situation of Proposition 1.3.20 and define* $\phi_k \colon H_r(H_k(D) \otimes_Q B(Q) \otimes K^u) \to H_{k+r}(\bigcup \mathcal{A})$ *analogously. Then we have* $\phi_k(x) \bullet \phi_l(y) = 0$, *unless* $|x| = |y| = 0$, *in which case the product is determined by*

$$\phi_k([a \otimes \langle p \rangle \otimes m]) \bullet \phi_l([b \otimes \langle q \rangle \otimes m']) = \phi_{k+l-n}([(a \bullet b) \otimes \langle p \wedge q \rangle \otimes mm']).$$

Proof. Assume that $|x| > 0$. As in the preceding proof, x maps to zero in $H(H(D) \otimes_Q B(Q) \otimes_Q R)$. Since the isomorphisms in (1.8) are natural with respect to K, the diagram

$$\begin{array}{ccc} H(H(D) \otimes_Q B(Q) \otimes_Q R^u) & \longrightarrow & H(H(D) \otimes_Q B(Q) \otimes_Q R) \\ \downarrow \phi & & \downarrow \\ H_*(\bigcup \mathcal{A}) & \xrightarrow{\quad i_* \quad} & H_*(X) \end{array}$$

commutes. It follows that $i_*(\phi(x)) = 0$. From this it follows that $\phi(x) \in \operatorname{im}\Big(H_*(X, \bigcup \mathcal{A}) \xrightarrow{\partial} H_*(\bigcup \mathcal{A})\Big)$ and therefore $\phi(x) \bullet \phi(y) = 0$, since $\phi(x) \bullet \phi(y) \in \operatorname{im}\Big(H_*(\bigcup \mathcal{A}, \bigcup \mathcal{A}) \xrightarrow{\partial} H_*(\bigcup \mathcal{A})\Big)$.

From the commutativity of (1.5), it follows that

$$\phi_k([a \otimes \langle p \rangle \otimes m]) \bullet \phi_l([b \otimes \langle q \rangle \otimes m']) = mi_*^p(a) \bullet m' i_*^q(b) = \\ = mm' i_*^{p \wedge q}(a \bullet b) = \phi_{k+l-n}([(a \bullet b) \otimes \langle p \wedge q \rangle \otimes mm']),$$

where $i^p \colon D(p) \to X$ is the inclusion map. □

The intersection product in $H_*(X, \bigcup \mathcal{A})$ is much more interesting and will be our object of study in concrete classes of arrangements.

Chapter 2

Linear and related arrangements

Let V be an $(n+1)$–dimensional vector space over $\mathbb{K}$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. We consider a linear arrangement $\mathcal{A}$ in V, that is a finite set of proper linear subspaces of V. We define Q to be the intersection poset of $\mathcal{A}$, that is the set $\{\bigcap C : C \subset \mathcal{A}\}$ ordered by inclusion.

2.0.23 Definition. We denote the category of finite dimensional vector spaces over $\mathbb{K}$ and linear monomorphisms by $\mathfrak{V}$. We define $D^{\mathcal{A}} \in \mathfrak{V}^{Q^o}$ by $D^Q(q) := q$ and letting $D^{\mathcal{A}}(p \leftarrow q) \colon D^{\mathcal{A}}(p) \to D^{\mathcal{A}}(q)$ be the inclusion map.

2.0.24 Notation. For $u \in Q$, we set $d(u) := \dim_{\mathbb{K}} u - 1$. For $S \subset \mathbb{N}$, we set $Q_S := \{q \in Q \colon d(q) \in S\}$.

2.1 Homology and Homotopy

ZŽ–maps

We will subsequently construct ZŽ–maps involving arrangements associated with the linear arrangement $\mathcal{A}$. As a basis for these we will now construct a set of maps that could be called a linear ZŽ–map. All of these constructions will depend on choices of points in V with certain properties. The most fundamental case is the following.

2.1.1 Proposition. *There is a function x assigning to each $u \in Q$ a system $(x_j^u)_{0 \le j \le d(u)}$ of $d(u)+1$ vectors in u such that for all $k \in \mathbb{N}$, $u_0, \ldots, u_k \in Q$, with $u_0 < u_1 < \cdots < u_r$ and $\lambda = (\lambda_0, \ldots, \lambda_r) \in \Delta^r$, the system of vectors*

$$\left(\sum_{i=0}^{r} \lambda_i x_j^{u_i} \right)_{0 \le j \le d(u_0)}$$

is linearly independent.

Proof. We give a simple recursive construction, because similar ones will be important later on. Let Λ be a linear functional on V that vanishes on no element of $Q_{[0,n]}$ and set $H := \ker \Lambda$. By induction there are $(x_j^u)_{0 \le j < d(u)}$ with $x_j^u \in u \cap H$ such that for all $k, r \in \mathbb{N}$, $u_0 < u_1 < \cdots < u_r$, $\lambda \in \Delta^r$, the system

$\left(\sum_i \lambda_i x_j^{u_i}\right)_{0\leq j<d(u_0)}$ is linearly independent. (The case $u_0 = 0$ is trivial.) We now choose $x_{d(u)}^u \in u \setminus H$ for all u. Now, if $\mu \in \mathbb{K}^{d(u_0)+1}$ with $\sum_j \mu_j \sum_i \lambda_i x_j^{u_i} = 0$, then $\mu_{d(u_0)}\lambda_0 x_{d(u_0)}^{u_0} \in H$ and therefore $\mu_{d(u_0)}\lambda_0 = 0$. If $\lambda_0 = 0$ then $\mu = 0$ by linear independence of $\left(\sum_{i=1}^r \lambda_i x_j^{u_i}\right)_{0\leq j\leq d(u_0)}$. If $\mu_{d(u)} = 0$, then $\mu = 0$ by linear independence of $\left(\sum_{i=0}^r \lambda_i x_j^{u_i}\right)_{0\leq j<d(u_0)}$. □

It will be important whether the space of possible choices of points is connected. One such case is the following.

2.1.2 Proposition. *For $\mathbb{K} = \mathbb{C}$, the set of all functions x as in Proposition 2.1.1, considered as a subspace of the affine space $\prod_{u\in Q} u^{d(u)+1}$ contains a non-empty Zariski-open set and is hence path-connected.*

Proof. The complement of the set is contained in the union of the sets

$$\left\{x\colon \text{Ex. } 0 \neq \lambda \in \mathbb{C}^{r+1} \text{ and } 0 \neq \mu \in \mathbb{C}^{d(u_0)+1} \text{ s.t. } \sum_{i=0}^{k}\sum_{j=0}^{d(u_0)} \lambda_i \mu_j x_j^{u_i} = 0\right\}$$

for all chains $u_0 < u_1 < \cdots < u_r$. Since the defining equations are homogenous in λ and μ, these sets are algebraic by the main theorem of elimination theory[Sha94, I.5, Thm 3]. Since the affine space is irreducible, it will be sufficient to show that each of these sets has non-empty complement. So we fix a chain $u_0 < u_1 < \cdots < u_k$. We choose a basis $(e_l)_{l=0,\dots,n}$ of V such that $e_l \in u_i$ for $l \leq d(u_0) + l$ and set $x_j^i := e_{i+j}$. Now if $\sum_j \sum_i \mu_j \lambda_i x_j^i = 0$ then $\sum_i \lambda_i \mu_{s-i} = 0$ for all s, and it follows that $\lambda = 0$, $\mu = 0$. □

2.1.3 Definition. Given a function x as in Proposition 2.1.1, we define functions f^p for $p \in Q$ by

$$\begin{aligned} f^p\colon \mathbb{K}^{d(p)+1} \times \Delta[p, V] &\to V, \\ \left((\mu_0, \dots, \mu_{d(p)}), \sum_{i=0}^{r} \lambda_i u_i\right) &\mapsto \sum_{j=0}^{d(p)}\sum_{i=0}^{r} \mu_j \lambda_i x_j^{u_i}. \end{aligned} \tag{2.1}$$

2.1.4 Proposition. *For any $y \in \Delta[p, q]$, the map $f^p(\bullet, y)$ is a linear injection into $q \subset V$ (and so necessarily an isomorphism, if $p = q$), and for $p \leq q$, the maps f^p and f^q agree on $\mathbb{K}^{d(p)+1} \times \Delta[q, V]$ where we identify $\mathbb{K}^{d(p)+1}$ with $\{\mu \in \mathbb{K}^{d(q)+1}\colon \mu_i = 0 \text{ for } i > d(p)\}$.* □

This proposition can be formulated as the maps f^p forming a commutative diagram as (1.6) with V for X and the functor F^Q now to be defined for E. With this definition we follow [WZŽ99, p. 141]

2.1.5 Definition. We define $F^Q \in \mathfrak{V}^{Q^o}$ by $F^Q(q) := \mathbb{K}^{d(q)+1}$ and letting $F^Q(p \leftarrow q)\colon \mathbb{K}^{d(p)+1} \to \mathbb{K}^{d(q)+1}$ be the standard inclusion.

Spherical arrangements

We now define and study the *spherical arrangement* $\mathbb{S}\mathcal{A}$ associated to the linear arrangement $\mathcal{A}$. This is done in some detail mainly to demonstrate the methods at our hands. The homological results will be obtained again later on as special cases of affine arrangements.

2.1.6 Definition. For a finite dimensional $\mathbb{K}$–vector space u, we set $\mathbb{S}(u) := (u \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation identifying x and y if there exists $\lambda > 0$ with $x = \lambda y$. If u is equipped with a norm, we can alternatively set $\mathbb{S}(u) := \{x \in u\colon |x| = 1\}$. For a linear injection $f\colon u \to v$ we define $\mathbb{S}(f)\colon \mathbb{S}(u) \to \mathbb{S}(v)$ by $[x] \mapsto [f(x)]$, making $\mathbb{S}$ into a functor from $\mathfrak{V}$ to $\mathfrak{T}op$.

2.1.7 Definition. We define the *spherical arrangement* $\mathbb{S}\mathcal{A}$ derived from $\mathcal{A}$ by $\mathbb{S}\mathcal{A} := \{\mathbb{S}A\colon A \in \mathcal{A}\}$. The union $\bigcup \mathbb{S}\mathcal{A}$ of $\mathbb{S}\mathcal{A}$ is usually called the *link of* $\mathcal{A}$ and the complement $\mathbb{S}V \setminus \bigcup \mathbb{S}\mathcal{A}$ is homotopy equivalent the the complement $V \setminus \bigcup \mathcal{A}$ of $\mathcal{A}$.

We will describe the homotopy type and the homology groups of the link of $\mathcal{A}$, as well as the cohomology groups of the complement, which are isomorphic to $H_*(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A})$ via Poincaré duality.

2.1.8 Definition. From the functions $f^p\colon \mathbb{K}^{d(p)+1} \times \Delta[p, V] \to V$ we derive functions $\mathbb{S}(f^p)\colon \mathbb{S}(\mathbb{K}^{d(p)+1}) \times \Delta[p, V] \to \mathbb{S}(V)$ by $\mathbb{S}(f^p)(x, y) := \mathbb{S}(f^p(\bullet, y))(x)$. More explicitly

$$\begin{aligned} &\mathbb{S}(f^p)\colon \mathbb{S}(\mathbb{K}^{d(p)+1}) \times \Delta[p, V] \to \mathbb{S}(V) \\ &\left([(\mu_0, \ldots, \mu_{d(p)})], \sum_{i=0}^{r} \lambda_i u_i\right) \mapsto \left[\sum_{j=0}^{d(p)} \sum_{i=0}^{r} \mu_j \lambda_i x_j^{u_i}\right]. \end{aligned} \tag{2.2}$$

We note that $\mathbb{S}(\mathbb{R}^{d(p)+1}) \approx \mathbb{S}^{d(p)}$ and $\mathbb{S}(\mathbb{C}^{d(p)+1}) \approx \mathbb{S}^{2d(p)+1}$.

2.1.9 Proposition. *The maps $\mathbb{S}(f^p)$ induce a homotopy equivalence*

$$\operatorname{hcolim} \mathbb{S}(F^{Q_{[0,n)}}) \xrightarrow{\simeq} \bigcup \mathbb{S}\mathcal{A}.$$

Proof. By Proposition 2.1.4 the conditions of Proposition 1.3.5 are satisfied. □

This shows that the homotopy type of $\bigcup \mathbb{S}\mathcal{A}$ depends on the intersection poset Q equipped with the dimension function d only. However, a different description of this homotopy type is more usual.

2.1.10 Proposition. *Let $M \in \mathfrak{T}op^{{Q^o}_{[0,n)}}$ be defined by $M(q) := \mathbb{S}(\mathbb{K}^{d(q)+1})$ and all non-identity morphisms being constant to the base-point. Then* $\operatorname{hcolim} M \simeq \bigcup \mathbb{S}\mathcal{A}$.

Proof. We want to compare the diagrams M and $\mathbb{S}(F^{Q_{[0,n)}})$. We have $M(q) = \mathbb{S}(F^{Q_{[0,n)}})(q)$ for all q, and the inclusion maps in $\mathbb{S}(F^{Q_{[0,n)}})$ are homotopic to the constant maps in M. These homotopies can be arranged to form a ZŽ–map.

Let $H\colon \mathbb{S}^\infty \times I \to \mathbb{S}^\infty$ be a base-point preserving homotopy from the identity to the constant map satisfying $H[\mathbb{S}^k \times I] \subset \mathbb{S}^{k+1}$ for all k. The latter condition can be achieved for example by cellular approximation. We define maps f_p^q for $p \leq q$, $p, q \in Q_{[0,n)}$ by

$$\begin{aligned} f_p^q\colon M(p) \times \Delta[p,q] &\to \mathbb{S}(F^{Q_{[0,n)}})(q) \\ \left(x, \sum_{i=0}^r \lambda_i u_i\right) &\mapsto \begin{cases} H(x, 1-\lambda_0), & u_0 = p, \\ *, & u_0 > p. \end{cases} \end{aligned}$$

For $p < p'$, we have $f_{p'}^q(M(p \leftarrow p')(x), t) = H(*, t) = * = H(x,1) = f_p^q(x,t)$. For $q < q'$, $f_p^{q'}(x,t) = F^Q(f_p^q(x,t))$. Therefore, the maps f_p^q induce a map of ${Q^o}_{[0,n)}$-diagrams $M \times_Q EQ \to F^{Q_{[0,n)}}$. Since f_p^p is a homeomorphism, this map of diagrams is a ZŽ–map. By Proposition 1.1.20 it induces a homotopy equivalence $\operatorname{hcolim} M \xrightarrow{\simeq} \operatorname{hcolim} \mathbb{S}(F^{Q_{[0,n)}})$. □

2.1.11 Remark. Of course, it would have been just as easy to give a ZŽ–map $M \times_Q EQ \to \mathbb{S}(D^{\mathcal{A}})$ directly.

We turn to homology calculations.

2.1.12 Proposition. *The map*

$$\begin{aligned} S(\mathbb{S}(F^Q)) \otimes_Q B(Q) &\to S(\mathbb{S}(D^{\mathcal{A}})) \\ c \times \big(p \leftarrow q_0 \leftarrow \ldots \leftarrow q_n \leftarrow p'\big) &\mapsto \mathbb{S}(f^p)_*(c \times \langle q_0, \ldots, q_n \rangle) \end{aligned} \tag{2.3}$$

is a ZŽ–map and therefore induces isomorphisms

$$H(S(\mathbb{S}(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \xrightarrow{\cong} H_*\left(\bigcup \mathbb{S}\mathcal{A}\right) \tag{2.4}$$

and

$$H(S(\mathbb{S}(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \xrightarrow{\cong} H_*\left(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A}\right). \tag{2.5}$$

Proof. This is an application of Proposition 1.3.12. The conditions of that Proposition are met because of Proposition 2.1.4. □

This argument is not specific to the functor $\mathbb{S}\colon \mathfrak{V} \to \mathfrak{T}op$ and we will investigate similar functors later on.

Proposition 2.1.12 already describes the homology of the link and, via Poincaré duality, the cohomology of the complement in terms of the intersection poset Q and the dimension function d. We will now simplify this description.

2.1.13 Definition and Proposition. *Let* $e_n = [c_n]$ *be a generator of* $\tilde{H}_n(\mathbb{S}^n)$ *and* $b_n \in S_{n+1}(\mathbb{S}^{n+1})$ *with* $\mathfrak{d} b_n = c_n$*. Also let* $a := \langle 1 \rangle \in C_0(S^0)$*. The map*

$$\begin{aligned} H(S(\mathbb{S}(F^Q))) \otimes_Q B(Q) &\to S(\mathbb{S}(F^Q)) \\ [a] \otimes \big(p \leftarrow q_0 \leftarrow \ldots \leftarrow q_n \leftarrow p'\big) &\mapsto \begin{cases} a, & n = 0 \\ 0, & n > 0 \end{cases} \\ e_k \otimes \big(p \leftarrow q_0 \leftarrow \ldots \leftarrow q_n \leftarrow p'\big) &\mapsto \begin{cases} c_k, & n = 0, p = q_0 \\ (-1)^{k+1} b_k, & n = 1, p = q_0 < q_1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.6}$$

is a ZŽ–map.

Proof. The map $h_r \colon H(\mathbb{S}(\mathbb{K}^{r+1})) \to S(\mathbb{S}(\mathbb{K}^{r+1}))$ mapping e_k to c_k and $[a]$ to a is a chain homotopy equivalence. If $r < s$ and $i \colon \mathbb{S}(\mathbb{K}^{r+1}) \to \mathbb{S}(\mathbb{K}^{s+1})$ is the inclusion map, then $e_k \mapsto b_k$ is a chain homotopy from $h_s \circ H(i)$ to $S(i) \circ h_r$. The map (2.6) can now be seen as a special case of that from Proposition 1.2.13 □

2.1.14 Proposition. *There are isomorphisms*

$$H(H(S(\mathbb{S}(F^Q))) \otimes_Q B(Q) \otimes_Q K^u) \xrightarrow{\cong} H_*\left(\bigcup \mathbb{S}\mathcal{A}\right) \tag{2.7}$$

and

$$H(H(S(\mathbb{S}(F^Q))) \otimes_Q B(Q) \otimes_Q K^p) \xrightarrow{\cong} H_*\left(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A}\right). \tag{2.8}$$

Proof. By Proposition 1.2.15, the ZŽ–maps from Proposition 2.1.12 and (2.6) combine to yield ZŽ–maps inducing the desired isomorphisms. □

2.1.15 Proposition. *For* $\mathbb{K} = \mathbb{R}$ *the above maps induce isomorphisms*

$$\begin{aligned} H_*\left(\bigcup \mathbb{S}\mathcal{A}\right) &\cong H_*(\Delta(Q_{[0,n)})) \oplus \bigoplus_{q \in Q_{[0,n)}} H_*(\Delta[q,V), \Delta(q,V))[-d(q)], \\ H_*\left(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A}\right) &\cong H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}) \\ &\quad \oplus \bigoplus_{q \in Q_{[0,n]}} H_*(\Delta[q,V], \Delta[q,V) \cup \Delta(q,V])[-d(q)]. \end{aligned}$$

Proof. The Q^o–diagram $H(\mathbb{S}(F^Q))$ decomposes as

$$H(\mathbb{S}(F^Q)) = P \oplus \bigoplus_{q \in Q_{[0,n]}} O_q \tag{2.9}$$

with P corresponding to the class of a 0-simplex and O_q to a generator of $\tilde{H}_{d(q)}(\mathbb{S}^{d(q)})$, i.e.

$$P(q) \cong \begin{cases} R, & 0 \le d(q) \le n, \\ 0, & d(q) = -1, \end{cases} \tag{2.10}$$

and all morphism the identity where possible, and

$$O_q(p) \cong \begin{cases} R[-d(q)], & p = q, \\ 0, & p \neq q. \end{cases} \tag{2.11}$$

Now

$$\begin{aligned} H(P \otimes_Q B(Q) \otimes_Q K^u) &\cong H_*(\Delta Q_{[0,n)}), \\ H(P \otimes_Q B(Q) \otimes_Q K^p) &\cong H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}), \\ H(Q_q \otimes_Q B(Q) \otimes_Q K^u) &\cong H_*(\Delta[q, V), \Delta(q, V))[-d(q)], \\ H(Q_q \otimes_Q B(Q) \otimes_Q K^p) &\cong H_*(\Delta[q, V], \Delta[q, V) \cup \Delta(q, V])[-d(q)] \end{aligned}$$

proving the proposition. □

2.1.16 Proposition. *For* $\mathbb{K} = \mathbb{R}$

$$\tilde{H}_*\left(\bigcup \mathbb{S}\mathcal{A}\right) \cong \bigoplus_{q \in [\perp, V)} \tilde{H}_*(\Delta(q, V))[-(d(q) + 1)], \tag{2.12}$$

$$H_*\left(\mathbb{S}V, \bigcup \mathbb{S}\mathcal{A}\right) \cong \mathbb{Z}[-n] \oplus \bigoplus_{q \in [\perp, V)} \tilde{H}_*(\Delta(q, V))[-d(q) - 2]. \tag{2.13}$$

Proof. Since $[q, V)$ has, for $q < V$, the minimum q, $\Delta[q, V)$ is acyclic and $H_k(\Delta[q, V), \Delta(q, V)) \cong \tilde{H}_{k-1}(\Delta(q, V))$. If $\perp \in Q_{[0,n)}$, then $Q_{[0,n)}$ has $\perp$ as its minimum and $\tilde{H}_*(\Delta Q_{[0,n)}) = 0$. Otherwise, $\tilde{H}_*(\Delta Q_{[0,n)}) = \tilde{H}_*(\Delta(\perp, V))$. This proves the first isomorphism.

$H_k(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}) \cong \tilde{H}_{k-1}(\Delta_{[0,n)})$ and for $q < V$

$$H_k(\Delta[q, V], \Delta[q, V) \cup \Delta(q, V]) \cong H_{k-1}(\Delta[q, V), \Delta(q, V)),$$

and these have been dealt with in the preceding paragraph. Also

$$H_*(\Delta[V, V], \Delta[V, V) \cup \Delta(V, V])[-d(V)] = H_*(\Delta\{V\})[-n] \cong \mathbb{Z}[-n],$$

completing the proof of the second isomorphism. □

Projective arrangements

We now come to our main object of study, projective arrangements.

For the functor P associating to a $\mathbb{K}$–vector–space u the projective space Pu we proceed just as for the functor $\mathbb{S}$ before.

2.1.17 Proposition. *The maps* $P(f^p)$ *induce a homotopy equivalence*

$$\operatorname{hcolim} P(F^{Q_{[0,n)}}) \xrightarrow{\simeq} \bigcup P\mathcal{A}.$$

Proof. Again, by Proposition 2.1.4 the conditions of Proposition 1.3.5 are satisfied. □

2.1.18 Remark. This is the same description of the homotopy type of a projective arrangement as in [WZŽ99, Prop. 5.9]. Our methods are very similar and also applicable to the other cases considered there. Our approach which makes ZŽ–maps more central has the advantage of explicitly constructing a homotopy equivalence.

2.1.19 Proposition. *The map*

$$\begin{gathered} S(P(F^Q)) \otimes_Q B(Q) \to S(P(D^{\mathcal{A}})) \\ c \times \left(p \leftarrow q_0 \leftarrow \ldots \leftarrow q_n \leftarrow p'\right) \mapsto P(f^p)_*(c \times \langle q_0, \ldots, q_n \rangle) \end{gathered} \tag{2.14}$$

is a ZŽ–map and therefore induces isomorphisms

$$H(S(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \xrightarrow{\cong} H_*\left(\bigcup P\mathcal{A}\right) \tag{2.15}$$

and

$$H(S(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \xrightarrow{\cong} H_*\left(PV, \bigcup P\mathcal{A}\right). \tag{2.16}$$

Proof. Just as Proposition 2.1.12 □

We simplify this description, starting with the most difficult case. The following result will actually also hold in the other cases treated afterwards.

2.1.20 Proposition. *Let* $\mathbb{K} = \mathbb{R}$ *and* $R = \mathbb{Z}$*. Then*

$$H_*\left(\bigcup P\mathcal{A}\right) \cong H(H(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \tag{2.17}$$

and

$$H_*\left(PV, \bigcup P\mathcal{A}\right) \cong H(H(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p). \tag{2.18}$$

Proof. Let $M \in \mathfrak{d}\mathfrak{A}b^{Q^o}$ be defined by

$$M(q)_k = \begin{cases} \mathbb{Z}, & 0 \le k \le d(q) \\ 0, & \text{otherwise,} \end{cases} \qquad M(q)_{k+1} \xrightarrow{\mathfrak{d}} M(q)_k = \begin{cases} 2, & 0 \le k < d(q),\ k \text{ odd,} \\ 0, & \text{otherwise,} \end{cases}$$

and the morphisms in M the identity where possible. It follows from the usual cell decomposition of $\mathbb{R}P^k$ with one cell per dimension that there exists a chain map of Q^o-diagrams $M \to F^Q$ inducing an isomorphism in homology. M allows a direct sum decomposition $M = \bigoplus_i M^i$ with $M^i_k := M_k$ for $k \in \{2i-1, 2i\}$ and $M^i_k := 0$ otherwise. The homology of M^i is nonzero in a single dimension only, hence Proposition 1.2.18 is applicable and yields the result together with Proposition 2.1.19. □

Possibly looking less attractive, these are as explicit combinatorial formulas as those derived in Proposition 2.1.24 for other cases. Nevertheless, for calculating the groups in question, the following may be helpful.

2.1.21 Proposition. *There are isomorphisms*

$$\begin{aligned} H(H_0(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) &\cong H_*(\Delta Q_{[0,n)}), \\ H(H_{2i}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) &\cong 0 \quad \text{for } i > 0, \end{aligned}$$

and there is a long exact sequence

$$\begin{aligned} \cdots \to H_k(\Delta Q_{[2i+2,n)}) &\xrightarrow{2j_*} H_k(\Delta Q_{[2i+1,n)}) \\ &\to H_k(H_{2i+1}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \to H_{k-1}(\Delta Q_{[2i+2,n)}) \xrightarrow{2j_*} \cdots \end{aligned}$$

for $i \geq 0$*, where* j *is the inclusion map. Similarly, there are isomorphisms*

$$\begin{aligned} H(H_0(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) &\cong H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}), \\ H(H_{2i}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) &\cong 0 \quad \text{for } i > 0, \end{aligned}$$

and there is a long exact sequence

$$\begin{aligned} \cdots \to H_k(\Delta Q_{[2i+2,n]}, \Delta Q_{[2i+2,n)}) &\xrightarrow{2j_*} H_k(\Delta Q_{[2i+1,n]}, \Delta Q_{[2i+1,n)}) \to \\ H_k(H_{2i+1}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) &\to H_{k-1}(\Delta Q_{[2i+2,n]}, \Delta Q_{[2i+2,n)}) \xrightarrow{2j_*} \cdots \end{aligned}$$

for $i \geq 0$.

Proof. The parts dealing with $H(H_{2i+1}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K)$ for $K = K^u$ or $K = K^p$ are the interesting ones. With $M_k \in \mathfrak{Ab}^{Q^o}$ as in the preceding proof, there is a short exact sequence $0 \to M_{2i+2} \xrightarrow{2} M_{2i+1} \to H_{2i+1}(P(F^Q)) \to 0$. Since $B(Q) \otimes_Q K$ is free, this induces a short exact sequence

$$\begin{aligned} 0 \to M_{2i+2} \otimes_Q B(Q) \otimes_Q K &\xrightarrow{2} M_{2i+1} \otimes_Q B(Q) \otimes_Q K \\ &\to H_{2i+1}(P(F^Q)) \otimes_Q B(Q) \otimes_Q K \to 0 \end{aligned}$$

and the long exact sequences stated in the proposition. □

The following cases are easier.

2.1.22 Definition and Proposition. *Let* $\mathbb{K} = \mathbb{C}$*, or let* $\mathbb{K} = \mathbb{R}$ *and* $R = \mathbb{Z}_2$*. Let* $e_n = [c_n]$ *be a generator of* $H_{2n}(\mathbb{C}P^n)$ *or* $H_n(\mathbb{R}P^n; \mathbb{Z}_2)$ *respectively. For* $i \geq n$ *we define* $e_n^i \in H_*(\mathbb{K}P^i; R)$ *by* $e_n^i = [c_n]$*. The maps*

$$\begin{aligned} H(P(F^Q(u))) &\to S(P(F^Q(u))), \\ e_n^i &\mapsto c_n \end{aligned} \tag{2.19}$$

for $u \in Q$ *induce isomorphisms in homology and form a chain map of* Q^o*–diagrams* $H(S(P(F^Q))) \to S(P(F^Q))$*.* □

2.1.23 Proposition. *For $\mathbb{K} = \mathbb{C}$, as well as for $\mathbb{K} = \mathbb{R}$ and $R = \mathbb{Z}_2$, there are isomorphisms*

$$H(H(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^u) \xrightarrow{\cong} H_*\left(\bigcup P\mathcal{A}\right) \tag{2.20}$$

and

$$H(H(P(F^Q)) \otimes_Q B(Q) \otimes_Q K^p) \xrightarrow{\cong} H_*\left(PV, \bigcup P\mathcal{A}\right). \tag{2.21}$$

Proof. The composition of (2.19) with the ZŽ–map (2.14) again yields a ZŽ–map and can therefore be substituted for (2.14) in Proposition 2.1.19. □

2.1.24 Proposition. *The above maps induce isomorphisms*

$$H_*\left(\bigcup P\mathcal{A}; \mathbb{Z}_2\right) \cong \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n)}; \mathbb{Z}_2)[-k], \tag{2.22}$$

$$H_*\left(PV, \bigcup P\mathcal{A}; \mathbb{Z}_2\right) \cong \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}; \mathbb{Z}_2)[-k] \tag{2.23}$$

for $\mathbb{K} = \mathbb{R}$, and

$$H_*\left(\bigcup P\mathcal{A}\right) \cong \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n)})[-2k], \tag{2.24}$$

$$H_*\left(PV, \bigcup P\mathcal{A}\right) \cong \bigoplus_{k=0}^{n} H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-2k] \tag{2.25}$$

for $\mathbb{K} = \mathbb{C}$.

Proof. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and $R = \mathbb{Z}_2$, there is a direct sum decomposition

$$H(P(F^Q)) = \bigoplus_{k=0}^{n} O_k, \qquad O_k(q) \cong \begin{cases} R[-mk], & d(q) \geq k, \\ 0, & d(q) < k \end{cases} \tag{2.26}$$

with $m = \dim_{\mathbb{R}} \mathbb{K}$, and $O_k(q \leftarrow q') = \mathrm{id}$ for $d(q) \geq k$. We choose the isomorphism in such a way that $1 \in O_k(q)$ corresponds to the canonical generator of $H_{mk}(\mathbb{K}P^{d(q)})$. Now

$$H(O_k \otimes_Q B(Q) \otimes_Q K^u) \cong H_*(\Delta Q_{[k,n)})[-mk], \tag{2.27}$$

$$H(O_k \otimes_Q B(Q) \otimes_Q K^p) \cong H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-mk], \tag{2.28}$$

and the result follows. □

We assemble the different maps to get the more direct handle on these isomorphisms that we will need when discussing intersection products. For Section 2.3 this, together with the preceding proposition, is the main result of this section.

2.1.25 Definition and Proposition. *We define*

$$f^k\colon \mathbb{K}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \to \left(PV, \bigcup P\mathcal{A}\right)$$
$$\left([(\mu_0,\ldots,\mu_k)], \sum_{i=0}^{r} \lambda_i u_i\right) \mapsto \left[\sum_{j=0}^{k}\sum_{i=0}^{r} \mu_j \lambda_i x_j^{u_i}\right] \tag{2.29}$$

and

$$h_k\colon H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \to H_*\left(PV, \bigcup P\mathcal{A}\right),$$
$$H_*(\Delta Q_{[k,n)}) \to H_*\left(\bigcup P\mathcal{A}\right) \tag{2.30}$$
$$c \mapsto f_*^k([\mathbb{K}P^k] \times c),$$

where $[\mathbb{K}P^k]$ is the orientation class of $\mathbb{K}P^k$, over $\mathbb{Z}_2$ in case $\mathbb{K} = \mathbb{R}$. Then all of the isomorphisms from Proposition 2.1.24 are given by $\sum_{k=0}^{n} h_k$.

For $\mathbb{K} = \mathbb{C}$, the maps h_k do not depend on the choice of the function x.

Proof. It is trivial to check that $\sum_k h_k$ is indeed the composition of the maps (2.14), (2.19), and (2.26). The non-dependence on the choice of x follows from Proposition 2.1.2. □

2.1.26 Remark. The calculations regarding the homology of the projective arrangements could equally well been carried out using the power set of $\mathcal{A}$ instead of the intersection poset, i.e., in the words of Remark 1.3.8, using naive resolutions instead of economical ones. This is because the key fact that all maps in the diagram $H(P(F^Q))$ are injective (assuming $\mathbb{K} = \mathbb{C}$ or $R = \mathbb{Z}_2$) still holds when Q denotes the power set of $\mathcal{A}$. For spherical arrangements however, naive resolutions would have been less appropriate, because the fact that all non-identity homomorphisms in the diagram $H_k(\mathbb{S}(F^Q))$ are zero for $k > 0$ would be lost, because there are non-identity isomorphisms in F^Q when Q denotes the power set of $\mathcal{A}$, so extra care has to be taken.

Affine arrangements

Let H be a hyperplane in V. We will investigate the relationship between the projective arrangement $P\mathcal{A}$, the projective arrangement induced on PH, and the arrangement induced on the affine space $PV \setminus PH$.

We set $\mathcal{A}^H := \{A \cap H\colon A \in \mathcal{A}\}$ and denote the intersection poset of $\mathcal{A}^H$ by Q^H.

We also set $\bar{Q} := Q \setminus Q^H = \{u \in Q\colon u \not\subset H\}$. This is the poset of non-empty intersections of the affine arrangement in $PV \setminus PH$.

2.1.27 Definition. We call the arrangement $\mathcal{A}$ a ≥ 2*–arrangement*, if $u < v$ implies $d(v) - d(u) \geq 2$ for all $u, v \in Q$.

2.1.28 Notation. Let X be a compact m-manifold and $A \subset X$ a closed subset. If $X \setminus A$ is orientable and such an orientation is chosen, we denote by $[X, A]$ the corresponding orientation class in $H_m(X, A)$.

2.1.29 Definition and Proposition. *We consider for $u \in \bar{Q}$ systems of vectors $b_0^u, \dots, b_{d(u)-1}^u \in u \cap H$, $x_u^u \in u$, $x_v^u \in v \cap H$ for $v > u$, such that*

$$\begin{aligned} f^u \colon \mathbb{K}P^{d(u)} \times \Delta[u, V] &\to PV \\ \left([\mu_0 : \dots : \mu_{d(u)}], \sum_{j=0}^{r} \lambda_j v_j \right) &\mapsto \left[\sum_{i=0}^{d(u)-1} \mu_i b_i^u + \mu_{d(u)} \sum_{j=0}^{r} \lambda_j x_{v_j}^u \right] \end{aligned} \tag{2.31}$$

is well defined, i.e. the term on the right hand side never equals zero. In particular the system of vectors $b_0^u, \dots, b_{d(u)-1}^u, x_u^u$ determines, in case $\mathbb{K} = \mathbb{R}$, an orientation of $(Pu, P(u \cap H))$ via the homeomorphism

$$\begin{aligned} \bar{f}^u \colon (\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}) &\to (Pu, P(u \cap H)) \\ [\mu_0 : \dots : \mu_{d(u)}] &\mapsto \left[\sum_{i=0}^{d(u)-1} \mu_i b_i^u + \mu_{d(u)} x_u^u \right] \end{aligned} \tag{2.32}$$

which is obtained by restricting f^u.

Such vectors exist, and the induced orientation can be prescribed in case $\mathbb{K} = \mathbb{R}$. There are induced maps

$$\begin{aligned} h_u \colon H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V]) &\to H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right), \\ H_*(\Delta[u, V), \Delta(u, V)) &\to H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right), \\ c &\mapsto f_*^u([\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] \times c). \end{aligned} \tag{2.33}$$

For $\mathbb{K} = \mathbb{C}$ these are independent of the choice of b^u and x^u. For $\mathbb{K} = \mathbb{R}$ and if $\mathcal{A}$ is a ≥ 2–arrangement, they depend only on the induced orientation of $(Pu, P(u \cap H))$. The maps

$$\bigoplus_{u \in \bar{Q}} H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V])[-md(u)] \xrightarrow{\sum_u h_u} H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right), \tag{2.34}$$

$$\bigoplus_{u \in \bar{Q}} H_*(\Delta[u, V), \Delta(u, V))[-md(u)] \xrightarrow{\sum_u h_u} H_*\left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right) \tag{2.35}$$

with $m = \dim_{\mathbb{R}} \mathbb{K}$ are isomorphisms.

2.1.30 Remark. If $d(\perp) \geq 0$, that is if $\perp \in \bar{Q}$, the affine arrangement is a (central) linear arrangement and $\bigcup P\mathcal{A} / (PH \cap \bigcup P\mathcal{A})$ is homeomorphic to the suspension of the link of that arrangement. Therefore, these isomorphisms generalize those of Proposition 2.1.15. The proof of Proposition 2.1.16 explains how to deal with the first summands in Proposition 2.1.15 when comparing it with the current proposition.

Proof. For the vectors b_i^u and x_v^u to define a map f^u, it suffices that for every chain $u \leq v_0 < \cdots < v_r$ the vectors $b_0^u, \ldots, b_{d(u)-1}^u, x_{v_0}^u, \ldots, x_{v_r}^u$ are linearly independent. This will be the case, if $(b_i^u)_i$ is a basis of $u \cap H$, $x_u^u \in u \setminus H$ and $x_v^u \in v \setminus w$ for $u \leq w < v$. The systems of vectors with this property form a nonempty Zariski-open set, which is therefore dense in the set of all allowed systems, and it is path-connected for $\mathbb{K} = \mathbb{C}$. For $\mathbb{K} = \mathbb{R}$ and if $\mathcal{A}$ is a ≥ 2-arrangement, it has at most four components, distinguished by the orientation of $u \cap H$ that $(b_i^u)_i$ defines and the component of $u \setminus H$ that contains x_u^u. Since replacing all of the b_i^u and x_v^u by their negatives does not change f^u, we may restrict x_u^u to one of the components of $u \setminus H$, and we see that the homotopy type of f^u depends only on the orientation induced on $(Pu, P(u \cap H))$.

We set $A(u) := (Pu, P(u \cap H))$ for $u \in \bar{Q}$. By Proposition 1.3.13 there are isomorphisms

$$H(S(A(\bar{D}^{\mathcal{A}})) \otimes_{\bar{Q}} K^p) \xrightarrow{\cong} H_* \left(PV, PH \cup \bigcup P\mathcal{A}\right) \tag{2.36}$$

and

$$H(S(A(\bar{D}^{\mathcal{A}})) \otimes_{\bar{Q}} K^u) \xrightarrow{\cong} H_* \left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right). \tag{2.37}$$

Let $O_u \in \mathfrak{d}\mathfrak{A}b^{\bar{Q}^o}$ be defined by

$$O_u(q) := \begin{cases} R[-md(u)], & u = q, \\ 0, & u \neq q \end{cases}$$

and $g_u \in \mathrm{Hom}_{\mathfrak{C}^o}(O_u \otimes_{\bar{Q}} B(\bar{Q}), S(A(\bar{D}^{\mathcal{A}})))$ by

$$1_R \otimes (u \leftarrow v_0 \leftarrow \cdots \leftarrow v_r \leftarrow u') \mapsto f_*^u(o_{d(u)} \times \langle v_0, \ldots, v_r \rangle),$$

where o_k is a relative cycle representing the orientation class of $(\mathbb{K}P^k, \mathbb{K}P^{k-1})$. This is well defined, because $f_*^u(o_{d(u)} \times \langle v_0, \ldots, v_r \rangle) = 0$ for $v_0 > u$. It is a chain map, because $f_*^u(\mathfrak{d} o_{d(u)} \times \langle v_0, \ldots, v_r \rangle) = 0$. We have

$$H(O_u \otimes_{\bar{Q}} B(\bar{Q}) \otimes_{\bar{Q}} K^p) \cong H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V])[-md(u)],$$
$$H(O_u \otimes_{\bar{Q}} B(\bar{Q}) \otimes_{\bar{Q}} K^u) \cong H_*(\Delta[u, V), \Delta(u, V))[-md(u)],$$

and the maps h_u equal $H(g_u \otimes \mathrm{id})$ composed with the above isomorphisms. The map $\sum_u \colon \bigoplus_u O_u \otimes_{\bar{Q}} B(\bar{Q}) \to S(A(\bar{D}^{\mathcal{A}}))$ is a ZŽ–map to a free diagram, therefore $\sum_u h_u$ is an isomorphism. □

That the direct sum decomposition $\bigoplus_u H_*(\Delta[u, V], \Delta[u, V) \cup \Delta(u, V])[-md(u)]$ is finer than the decomposition $\bigoplus_k H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-mk]$ can be regarded as a reason why products in affine arrangements are technically simpler than products in projective arrangements. In particular, there seems to be no direct generalization of the proof of Proposition 2.2.12 to a proof of Proposition 2.3.13.

We illustrate the connection between projective and affine arrangements by giving another description of the homomorphisms h_u. For simplicity, we only cover the case $\mathbb{K} = \mathbb{C}$ completely.

2.1.31 Definition and Proposition. *There is a function x as in Proposition 2.1.1 with the additional property that $x^u_j \in H$ for $j < \dim_{\mathbb{K}}(u \cap H)$. If such a function is used in definition of the map f^k from Definition 2.1.25, then f^k maps the subspace $\mathbb{K}P^k \times \Delta Q_{(k,n]} \cup \mathbb{K}P^{k-1} \times \Delta Q_{[k,n]}$ to PH and therefore induces maps*

$$\begin{aligned} \bar{h}_k \colon H_*(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]}) &\to H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right), \\ H_*(\Delta\bar{Q}_{[k,n)}, \Delta\bar{Q}_{(k,n)}) &\to H_*\left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right), \\ c &\mapsto f^k_*([\mathbb{K}P^k, \mathbb{K}P^{k-1}] \times c). \end{aligned} \tag{2.38}$$

Denoting the inclusion map $\Delta[u,V] \to \Delta Q_{[d(u),n]}$ by i^u, the maps

$$\sum_{u\in\bar{Q}_{\{k\}}} i^u_* \colon \bigoplus_{u\in\bar{Q}_{\{k\}}} H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V]) \xrightarrow{\cong} H_*(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]}) \tag{2.39}$$

and

$$\sum_{u\in\bar{Q}_{\{k\}}} i^u_* \colon \bigoplus_{u\in\bar{Q}_{\{k\}}} H_*(\Delta[u,V), \Delta(u,V)) \xrightarrow{\cong} H_*(\Delta\bar{Q}_{[k,n)}, \Delta\bar{Q}_{(k,n)}) \tag{2.40}$$

are isomorphisms.

For $\mathbb{K} = \mathbb{C}$ we have $h_u = \bar{h}_k \circ i^u_*$ and therefore the maps

$$\sum \bar{h}_k \colon \bigoplus_k H_*(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]})[-2k] \to H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right), \tag{2.41}$$

$$\bigoplus_k H_*(\Delta\bar{Q}_{[k,n)}, \Delta\bar{Q}_{(k,n)})[-2k] \to H_*\left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right) \tag{2.42}$$

are isomorphisms.

Proof. Slightly modifying the construction in the proof of Proposition 2.1.1, we first choose $(x^u_j)_{0\le j<\dim(u\cap H)}$ (applying the proposition to $\mathcal{A}^H$) and then choose $x^u_{d(u)} \in u \setminus H$ for all $u \in \bar{Q}$. As in that proof, x has the desired property.

The maps in (2.39) and (2.39) are easily seen to be isomorphisms either directly at the chain level or by a Mayer-Vietoris argument using $\Delta\bar{Q}_{[k,n]} = \bigcup_{u\in\bar{Q}_{\{k\}}} \Delta[u,V]$ and $\Delta[u,V] \cap \Delta[v,V] \subset \Delta\bar{Q}_{(k,n]}$ for $u, v \in \bar{Q}_{\{k\}}$, $u \ne v$.

Let $u \in \bar{Q}$. Any system of points (y^v_j) for $v \ge u$ and $0 \le j \le d(u)$ defines maps

$$\begin{aligned} g \colon (\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}) \times (\Delta[u,V], \Delta[u,V) \cup \Delta(u,V]) &\to \left(PV, PH \cup \bigcup P\mathcal{A}\right) \\ (\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}) \times (\Delta[u,V), \Delta(u,V)) &\to \left(\bigcup P\mathcal{A}, PH \cap \bigcup P\mathcal{A}\right) \\ \left([(\mu_0, \dots, \mu_k)], \sum_{i=0}^r \lambda_i u_i\right) &\mapsto \left[\sum_{j=0}^k \sum_{i=0}^r \mu_j \lambda_i x^{u_i}_j\right], \end{aligned}$$

if the right hand side is always well defined. For $\mathbb{K} = \mathbb{C}$ the space of such systems of points is again path connected and hence the homotopy type of g independent of the choice of such a system.

Setting $y^v_j := x^v_j$ gives such a system, and for this choice $g = f^k \circ i^u$. Setting $y^v_j := b^u_j$ for $j < d(u)$ and $y^v_{d(u)} := x^u_v$ with b and x the systems from Definition 2.1.29 gives another such system, and for this one $g = f^u$. In case $\mathbb{K} = \mathbb{C}$ it follows that $f^k \circ i^u \simeq f^u$ and $h_u = \bar{h}_k \circ i^u_*$. □

2.1.32 Remark. For a complex affine arrangement the isomorphism

$$H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right) \cong \bigoplus_k H_*(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]})[-2k],$$

and hence the isomorphism (2.34), can also be deduced combinatorially from the isomorphism (2.25) for projective arrangements by considering the arrangement $\mathcal{A}^+ := \mathcal{A} \cup \{H\}$. This is called an 'interesting exercise' in [GM88]. We sketch how to do this. To this end we denote the intersection poset of $\mathcal{A}^+$ by Q^+. For $0 \le k < n$, the simplical complex $\Delta(Q^+_{[k,n)} \setminus \bar{Q}_{\{k\}})$ is acyclic, since it contains the cone $\Delta\{q\colon q \le H, d(q) \ge k\}$ as a deformation retract. Therefore the first map in

$$\begin{aligned} H_*\big(\Delta Q^+_{[k,n]}, \Delta Q^+_{[k,n)}\big) &\xrightarrow{\cong} H_*\big(\Delta Q^+_{[k,n]}, \Delta Q^+_{[k,n)} \cup \Delta(Q^+_{[k,n]} \setminus \bar{Q}_{\{k\}})\big) \\ &\xleftarrow{\cong} H_*\big(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]}\big), \end{aligned}$$

which is induced by inclusion, is an isomorphism. The second map is also induced by inclusion and is an isomorphism by excision. The isomorphisms also hold in the trivial case $k = n$. This yields $H_*\left(PV, PH \cup \bigcup P\mathcal{A}\right) = H_*\left(PV, \bigcup P\mathcal{A}^+\right) \cong \bigoplus H_*\big(\Delta Q^+_{[k,n]}, \Delta Q^+_{[k,n)}\big)[-2k] \cong \bigoplus H_*\big(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{[k,n)} \cup \Delta\bar{Q}_{(k,n]}\big)[-2k]$.

2.1.33 Remark. If the arrangement $\mathcal{A}$ is in general position with respect to H, then $\bar{Q} = Q_{[0,n]}$ and $q \mapsto q \cap H$ induces isomorphisms $\eta\colon Q_{k+1} \to Q^H_k$. For $\mathbb{K} = \mathbb{C}$ it follows from the construction in Proposition 2.1.31 that the diagram

$$\begin{array}{ccc} H_i(\Delta Q^H_{[k,n-1]}, \Delta Q^H_{[k,n-1)}) & \xrightarrow{h^H_k} & H_{i+2k}(PH, \bigcup P\mathcal{A}^H) \\ \downarrow{\scriptstyle (\eta^{-1})_*} & & \downarrow{\scriptstyle \mathrm{incl}_*} \\ H_i(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) & \xrightarrow{h_k} & H_{i+2k}(PV, \bigcup P\mathcal{A}) \\ \downarrow{\scriptstyle \mathrm{incl}_*} & & \downarrow{\scriptstyle \mathrm{incl}_*} \\ H_i(\Delta\bar{Q}_{[k,n]}, \Delta\bar{Q}_{(k,n]} \cup \Delta\bar{Q}_{[k,n)}) & \xrightarrow{\bar{h}_k} & H_{i+2k}(PV, PH \cup \bigcup P\mathcal{A}) \end{array}$$

commutes, since $f^k \circ (\mathrm{id}_{\mathbb{C}P^k} \times \eta^{-1})$ is a suitable function f^k_H for the definition of h^H_k. Both columns are part of a long exact sequence of the form

$$H_j(B, A \cap B) \to H_j(X, B) \to H_j(X, A \cup B) \xrightarrow{\partial} H_{j-1}(B, A \cap B)$$

for an excisive triad $(X;A,B)$. Because of the naturality of the connecting homomorphism, the diagram

$$\begin{array}{ccc} H_i(\Delta\bar{Q}_{[k,n]},\Delta\bar{Q}_{(k,n]}\cup\Delta\bar{Q}_{[k,n)}) & \xrightarrow{\bar{h}_k} & H_{i+2k}(PV,PH\cup\bigcup P\mathcal{A}) \\ \downarrow{\scriptstyle \eta_*\circ\partial} & & \downarrow{\scriptstyle \partial} \\ H_{i-1}(\Delta Q^H_{[k,n-1]},\Delta Q^H_{[k,n-1)}) & \xrightarrow{h^H_k} & H_{i+2k-1}(PH,\bigcup P\mathcal{A}^H) \end{array}$$

also commutes. Analogous commutative diagrams exist for the long exact sequence of the pair $(\bigcup P\mathcal{A},\bigcup P\mathcal{A}^H)$. If the arrangement $\mathcal{A}$ is not in general position with respect to H, then the same commutative diagrams exist, but the construction of the left column requires more care.

2.1.34 Remark (Gysin sequence). We continue the preceding remark. We set $A := PV\setminus PH$ and denote by $\mathcal{A}^A := PB\setminus PH\colon B\in\mathcal{A}$ the induced arrangement in this affine space. Using Poincaré duality, which we denote by D, we switch to cohomology and obtain the following commutative diagram with exact columns.

$$\begin{array}{ccc} \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q^H_{[k,n-1]},\Delta Q^H_{[k,n-1)}) & \xrightarrow[\cong]{D\circ\sum_k h^H_k} & H^i(PH\setminus\bigcup P\mathcal{A}^H) \\ \downarrow{\scriptstyle (\eta^{-1})_*} & & \downarrow{\scriptstyle i^!} \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q_{[k,n]},\Delta Q_{[k,n)}) & \xrightarrow[\cong]{D\circ\sum_k h_k} & H^{i+2}(PV\setminus\bigcup P\mathcal{A}) \\ \downarrow{\scriptstyle \mathrm{incl}_*} & & \downarrow{\scriptstyle \mathrm{incl}^*} \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta\bar{Q}_{[k,n]},\Delta\bar{Q}_{(k,n]}\cup\Delta\bar{Q}_{[k,n)}) & \xrightarrow[\cong]{D\circ\sum_k \bar{h}_k} & H^{i+2}(A\setminus\bigcup\mathcal{A}^A) \\ \downarrow{\scriptstyle \eta_*\circ\partial} & & \downarrow \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i-1}(\Delta Q^H_{[k,n-1]},\Delta Q^H_{[k,n-1)}) & \xrightarrow[\cong]{D\circ\sum_k h^H_k} & H^{i+1}(PH\setminus\bigcup P\mathcal{A}^H) \end{array}$$

If the arrangement $\mathcal{A}^A$ is central, i.e. $\bigcap\mathcal{A}^A\neq\emptyset$, then there is a deformation retraction $\pi\colon PV\setminus\bigcup P\mathcal{A}\xrightarrow{\simeq} PH\setminus\bigcup P\mathcal{A}^H$ homotopy inverse to the inclusion i. Also $\bar{Q}=Q$. From the above we therefore obtain the following diagram.

$$\begin{array}{ccc} \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q_{[k+1,n]},\Delta Q_{[k+1,n)}) & \xrightarrow[\cong]{D\circ\sum h^H_k\circ\eta_*} & H^i(PH\setminus\bigcup P\mathcal{A}^H) \\ \downarrow{\scriptstyle \mathrm{incl}_*} & & \downarrow{\scriptstyle \smile\alpha} \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q_{[k,n]},\Delta Q_{[k,n)}) & \xrightarrow[\cong]{D\circ i_!\circ\sum_k h_k} & H^{i+2}(PH\setminus\bigcup P\mathcal{A}^H) \\ \downarrow{\scriptstyle \mathrm{incl}_*} & & \downarrow{\scriptstyle \pi^*} \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i}(\Delta Q_{[k,n]},\Delta Q_{(k,n]}\cup\Delta Q_{[k,n)}) & \xrightarrow[\cong]{D\circ\sum\bar{h}_k} & H^{i+2}(A\setminus\bigcup\mathcal{A}^A) \\ \downarrow{\scriptstyle \partial} & & \downarrow \\ \bigoplus_{k\geq 0} H_{2(n-k-1)-i-1}(\Delta Q_{[k+1,n]},\Delta Q_{[k+1,n)}) & \xrightarrow[\cong]{D\circ\sum h^H_k\circ\eta_*} & H^{i+1}(PH\setminus\bigcup P\mathcal{A}^H) \end{array}$$

Here $\alpha\in H^2(PH)$ is the canonical generator. Also note $H_*(\Delta Q_{[0,n]},\Delta Q_{[0,n)})=0$, since $Q_{[0,n]}$ has a minimal element, which explains the missing of a summand in the first row. The map $\pi\colon A\setminus\bigcup\mathcal{A}^A\to PH\setminus\bigcup P\mathcal{A}^H$ is a fibre bundle with fibre $\mathbb{C}\setminus\{0\}$, α is its Thom class, and the right column of the diagram is its Gysin sequence.

If we start with a linear arrangement $\mathcal{A}$ in V, pass to the arrangement $\mathcal{A}\times\mathbb{C}$ in $V\times\mathbb{C}$, and set $H:=V\times\{0\}$, then $P(\mathcal{A}\times\mathbb{C})^H$ will be $P\mathcal{A}$ and $(\mathcal{A}\times\mathbb{C})^A$ will be isomorphic to the original arrangement $\mathcal{A}$. The map π constructed above will correspond to the quotient map $V\setminus\bigcup\mathcal{A}\to PV\setminus\bigcup P\mathcal{A}$, so that we have just described its cohomology Gysin sequence completely combinatorially. Just the isomorphism in the second row is possibly not as explicit as we would like it. This, however, will be remedied in a minute, when we study the intersection with the hyperplane H to obtain $i_!\circ h_k=h^H_{k-1}\circ\eta_*$.

Intersecting with a hyperplane

We will describe the map on the homology of an affine or a projective arrangement given by intersecting with a hyperplane, that is the transfer map of the inclusion of the hyperplane. In contrast to the material covered so far in this chapter, with the exception of the results mentioning cohomology, this depends on the projective or affine space in which the arrangement is contained being a manifold. The same is of course true for the intersection products treated later on, and the results obtained here will be the basis for the inductive step in the proof of the product formulas.

Let $\Lambda^H\colon V\to\mathbb{K}$ be a linear functional that vanishes on no element of $Q_{[0,2n]}$ and $H:=\ker\Lambda^H$. $\mathcal{A}$ induces an arrangement $\mathcal{A}^H:=\{A\cap H\colon A\in\mathcal{A}\}$ in H. If we denote the intersection poset of $\mathcal{A}^H$ by Q^H,

$$\begin{aligned}\eta\colon Q_{(0,n]}&\to Q^H_{[0,n-1]}\\ q&\mapsto q\cap H\end{aligned}$$

is an isomorphism lowering dimensions by one.

We consider the inclusion map $i\colon\ \big(PH,\bigcup P\mathcal{A}^H\big)\to(PV,\bigcup P\mathcal{A})$.

2.1.35 Proposition. *Let $\mathbb{K}=\mathbb{C}$. For $c\in H_*(\Delta Q_{[k,n]},\Delta Q_{[k,n)})$ we have*

$$i_!(h_k(c))=\begin{cases}h^H_{k-1}(\eta_*(c)), & k>0,\\ 0, & k=0\end{cases}$$

and in particular $\ker i_!=\operatorname{im}h_0$.

Proof. We first choose $(x^u_j)_{0\le j<k}$ with $x^u_j\in u\cap H$ satisfying the conditions of Definition 2.1.25 and therefore defining functions f^{k-1}_H and h^H_{k-1}. Now for each $u\in Q_{[k,n]}$ we choose $x^u_k\in u$ with $\Lambda^H(x^u_k)=1$. $(x^u_j)_{0\le j\le k}$ then also satisfies the conditions of Definition 2.1.25 and can be used to define f^k and h_k.

Indeed we calculate

$$\Lambda^H\left(\sum_{j=0}^{k}\sum_{i=0}^{r}\mu_j\lambda_i x_j^{u_i}\right)=\sum_{i=0}^{r}\mu_k\lambda_i=\mu_k.$$

First this implies that $f^k(x,y)\in H$ iff $x\in\mathbb{C}P^{k-1}\subset\mathbb{C}P^k$. In particular f^0 misses H, which proves that part of the proposition, and we now assume $k>0$. The equation also implies that for $x=[x_0:\dots:x_{k-1}]\in\mathbb{C}P^{k-1}$ and $y\in\Delta(Q_{[k,n]})$ the map $\mu\mapsto f^k([x_0:\dots:x_{k-1}:\mu],y)$ meets H transversally. Furthermore

$$\begin{aligned}f^k\left([\mu_0:\dots:\mu_{k-1}:0],\sum_{i=0}^{r}\lambda_i u_i\right)&=\left[\sum_{j=0}^{k-1}\sum_{i=0}^{r}\mu_j\lambda_i x_j^{u_i}\right]\\&=f_H^{k-1}\left([\mu_0:\dots:\mu_{k-1}],\sum_{i=0}^{r}\lambda_i\eta(u_i)\right),\end{aligned}$$

which proves the proposition as we will now show in more detail.

Let $\vartheta\in H^2(PV,PV\setminus PH)$ be the Thom class of (the normal bundle of) PH in PV, i.e. the class satisfying $\theta\frown[PV]=[PH]$. By the above calculations $(f^k)^*(\vartheta)\in H^2\left(\mathbb{C}P^k\times\Delta Q_{[k,n]},(\mathbb{C}P^k\setminus\mathbb{C}P^{k-1})\times\Delta Q_{[k,n]}\right)$ is the Thom class of $\mathbb{C}P^{k-1}\times\Delta Q_{[k,n]}$ in $\mathbb{C}P^k\times\Delta Q_{[k,n]}$ which equals the class $\alpha\times 1$ where $\alpha\in H^2(\mathbb{C}P^k,\mathbb{C}P^k\setminus\mathbb{C}P^{k-1})$ which is again a Thom class and maps to the canonical generator of $H^*(\mathbb{C}P^k)$. We finally calculate

$$\begin{aligned}i_!(h_k(c))&=\vartheta\frown h_k(c)\\&=\vartheta\frown f_*^k([\mathbb{C}P^k]\times c)\\&=f_*^k\left((f^k)^*(\vartheta)\frown([\mathbb{C}P^k]\times c)\right)\\&=f_*^k\left((\alpha\times 1)\frown([\mathbb{C}P^k]\times c)\right)\\&=f_*^k\left((\alpha\frown[\mathbb{C}P^k])\times(1\frown c)\right)\\&=f_*^k\left([\mathbb{C}P^{k-1}]\times c\right)\\&=h_{k-1}(\eta_*(c))\end{aligned}$$

as claimed. □

Turning to affine arrangements we let $I\subset V$ be another hyperplane and consider the inclusion map $i\colon\left(PH,P(I\cap H)\cup\bigcup P\mathcal{A}^H\right)\to(PV,PI\cup\bigcup P\mathcal{A})$.

2.1.36 Proposition. *Let $\mathcal{A}$ be a complex arrangement or a real ≥ 2-arrangement. Let $I\subset V$ be a hyperplane (at infinity) and $H\subset V$ a hyperplane in general position with respect to $\mathcal{A}\cup\{I\}$. Let $u\in\bar{Q}$, $c\in H_*(\Delta[u,V],\Delta[u,V)\cup\Delta(u,V])$ and*

$$h_u\colon H_*(\Delta[u,V],\Delta[u,V)\cup\Delta(u,V])\to H_*\left(PV,PI\cup\bigcup P\mathcal{A}\right)$$

be the homomorphism from Definition 2.1.29. In case $d(u) > 0$*, we also consider the homomorphism*

$$h^H_{\eta(u)}\colon H_*(\Delta[\eta(u),\eta(V)],\Delta[\eta(u),\eta(V))\cup\Delta(\eta(u),\eta(V)]) \\ \to H_*\left(PH,(PH\cap PI)\cup\bigcup P\mathcal{A}^H\right).$$

In case $\mathbb{K}=\mathbb{R}$*, we assume the orientations of* $(Pu, PI\cap Pu)$ *and* $(P(u\cap H), P(u\cap H\cap I))$ *which are used in the definitions of* h_u *and* $h^H_{\eta(u)}$ *to be related by*

$$(\bar f_H^{\eta(u)})_*([\mathbb{R}P^{d(u)-1},\mathbb{R}P^{d(u)-2}]) = i_!\left(\bar f^u_*([\mathbb{R}P^{d(u)},\mathbb{R}P^{d(u)-1}])\right), \tag{2.43}$$

see (2.32)*.*

Considering the inclusion map $i\colon \left(PH, P(I\cap H)\cup\bigcup P\mathcal{A}^H\right)\to(PV, PI\cup\bigcup P\mathcal{A})$ *we have*

$$i_!(h_u(c)) = \begin{cases} h^H_{\eta(u)}(\eta_*(c)), & d(u)>0,\\ 0, & d(u)=0.\end{cases}$$

In particular $\ker i_! = \bigoplus_{u\in\bar Q_{\{0\}}} h_u$*.*

Proof. Let $\Lambda^I\colon V\to\mathbb{K}$ be a linear functional with $\ker\Lambda^I = I$.

For $d(u)=0$ we may assume that f^u and hence h_u is defined via points satisfying $\Lambda^H(x^u_v)=1$ for all $v\geq u$. Then

$$\Lambda^H\left(\sum_{i=0}^{d(u)-1}\mu_i b^u_i + \mu_{d(u)}\sum_{j=0}^{r}\lambda_j x^u_{v_j}\right) = \Lambda^H\left(\sum_{j=0}^{r}\lambda_j x^u_{v_j}\right) = 1 \tag{2.44}$$

and f^u misses H so that $i_!(h_u(c))=0$.

Now let $d(u)>0$. We may assume that f^u and hence h_u is defined via points satisfying $b^u_0,\dots,b^u_{d(u)-2}\in u\cap H\cap I$, $b^u_{d(u)-1}\in u\cap I$, $\Lambda^H(b^u_{d(u)-1})=1$, $x^u_u\in u\cap H$, $\Lambda^I(x^u_u)=1$, $x^u_v\in v\cap H\cap I$ for $v>u$. For these

$$\Lambda^H\left(\sum_{i=0}^{d(u)-1}\mu_i b^u_i + \mu_{d(u)}\sum_{j=0}^{r}\lambda_j x^u_{v_j}\right) = \mu_{d(u)-1}. \tag{2.45}$$

Hence, for $y\in\Delta[u,V]$ and $(\mu_0,\dots,\mu_{d(u)-2},\mu_{d(u)})\in\mathbb{K}^{d(u)}\setminus\{0\}$ the map $\mu\mapsto f^u\left([\mu_0:\dots:\mu_{d(u)-2}:\mu:\mu_{d(u)}],y\right)$ meets H transversally in $0\in\mathbb{K}$. Furthermore

$$\Lambda^I\left(\sum_{i=0}^{d(u)-1}\mu_i b^u_i + \mu_{d(u)}\sum_{j=0}^{r}\lambda_j x^u_{v_j}\right) = \begin{cases}0, & v_0>u,\\ \mu_{d(u)}\lambda_0, & v_0=u.\end{cases} \tag{2.46}$$

So $f^u(x,y)\in I$, iff $x\in\mathbb{K}P^{d(u)-1}$ or $y\in\Delta(u,V]$. We set $m:=\dim_{\mathbb{R}}\mathbb{K}$ and define $\vartheta\in H^m(PV\setminus PI, PV\setminus(PI\cup PH))$ to be the Thom class of $PH\setminus PI$ in $PV\setminus PI$, satisfying $\vartheta \frown [PV, PI] = [PH, PI \cap PH]$. It follows that $(f^u)^*(\vartheta) = \varepsilon\alpha \times 1$ with

$$\begin{aligned}
&\varepsilon \in \{+1, -1\}, \\
&\alpha \in H^m\left(\mathbb{K}P^{d(u)} \setminus \mathbb{K}P^{d(u)-1}, \mathbb{K}P^{d(u)} \setminus (\mathbb{K}P^{d(u)-1} \cup j[\mathbb{K}P^{d(u)-1}])\right), \\
&1 \in H^0((\Delta[u, V] \setminus \Delta(u, V))),
\end{aligned}$$

where α satisfies $\alpha \frown [\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] = [j[\mathbb{K}P^{d(u)-1}], j[\mathbb{K}P^{d(u)-2}]]$.

Ignoring $b^u_{d(u)-1}$, these points also define a map $f_H^{\eta(u)} \colon \mathbb{K}P^{d(u)-1} \times \Delta[\eta(u), \eta(V)] \to PH$ that induces h_u^H with

$$\begin{aligned}
f^u\left([\mu_0 : \cdots : \mu_{d(u)-2} : 0 : \mu_{d(u)}], \sum_{i=0}^{r} \lambda_i u_i\right) &= \\
&= \left[\sum_{i=0}^{d(u)-2} \mu_i b_i^u + \mu_{d(u)} \sum_{j=0}^{r} \lambda_j x_{v_j}^u\right] = \\
&= f_H^{\eta(u)}\left([\mu_0 : \cdots : \mu_{d(u)-2} : \mu_{d(u)}], \sum_{i=0}^{r} \lambda_i u_i\right), \quad (2.47)
\end{aligned}$$

i.e. $f^u(j(x), y) = f_H^{\eta(u)}(x, \eta(y))$. We can now calculate

$$\begin{aligned}
i_!(h_u(c)) &= \vartheta \frown h_u(c) \\
&= \vartheta \frown f_*^u([\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] \times c) \\
&= f_*^u\left((f^u)^*(\vartheta) \frown ([\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] \times c)\right) \\
&= f_*^u\left((\varepsilon\alpha \times 1) \frown ([\mathbb{K}P^{d(u)}, \mathbb{K}P^{d(u)-1}] \times c)\right) \\
&= \varepsilon f_*^u\left([j[\mathbb{K}P^{d(u)-1}], j[\mathbb{K}P^{d(u)-2}]] \times c\right) \\
&= \varepsilon (f_H^{\eta(u)})_*([\mathbb{K}P^{d(u)-1}, \mathbb{K}P^{d(u)-2}] \times \eta_*(c)) \\
&= \varepsilon h^H_{\eta(u)}(\eta_*(c)).
\end{aligned}$$

Similarly we find $i_!\left(\bar{f}_*^u([\mathbb{R}P^{d(u)}, \mathbb{R}P^{d(u)-1}])\right) = \varepsilon(\bar{f}_H^{\eta(u)})_*([\mathbb{R}P^{d(u)-1}, \mathbb{R}P^{d(u)-2}])$ and hence $\varepsilon = 1$ by comparison with (2.43) □

2.1.37 Remark. For affine ≥ 2-arrangements, Proposition 2.1.36 already almost solves the problem of describing the intersection product in combinatorial terms, as we will sketch now.

If $\mathcal{A}$ is an arrangement in X and $\mathcal{B}$ an arrangement in Y then the product $(X, \bigcup\mathcal{A}) \times (Y, \bigcup\mathcal{B})$ equals $(X \times Y, \bigcup(\mathcal{A} \times \mathcal{B}))$, where $\mathcal{A} \times \mathcal{B}$ is the arrangement in $X \times Y$ defined as $\{A \times Y \colon A \in \mathcal{A}\} \cup \{X \times B \colon B \in \mathcal{B}\}$. Combinatorial descriptions of the homology of $(X, \bigcup\mathcal{A})$ and of $(Y, \bigcup\mathcal{B})$ easily lead to combinatorial descriptions of the homology of the product and the cross product map. For projective arrangements we will carry this out partially in Proposition 2.3.2. For linear arrangements, see [dLS01, Prop. 4.1].

Having obtained a description of the cross product, to describe intersection products it then remains to describe the intersection with the diagonal, i.e. the transfer map of the diagonal map. The product of two affine arrangements is again an affine arrangement, hence the diagonal is an affine plane itself, and this can be done by applying Proposition 2.1.36 n times. The only hindrance would be that the diagonal is not in general position with respect to the product arrangement, since the original arrangement is not in general position with respect to itself. This problem could be dealt with using methods similar to those we will apply in the proof of Proposition 2.3.13 on page 67.

However, the product of two projective spaces is not again a projective space. Therefore, the description of intersection products in projective arrangements requires additional techniques.

2.2 Products

Statement of results

We remind the reader of the product $\hat{\times}$ from Definition 1.3.14.

2.2.1 Theorem. *Let $\mathcal{A}$ be a complex arrangement. For all $k, l \geq 0$ and all $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$ we have*

$$h_k(c) \bullet h_l(d) = \begin{cases} h_{k+l-n}(c \mathbin{\hat{\times}} d), & k+l \geq n, \\ 0, & k+l < n. \end{cases} \tag{2.48}$$

This is the main result of this work. Its proof will take up Section 2.3 and be completed on p. 68.

The statement of the corresponding result for affine arrangements will require some preparations, because we do not restrict ourselves to complex arrangements in this case.

The definition of the homomorphisms h_u in Definition 2.1.29 depends on the choice of a basis $(b_i^u)_i$ for $u \in \bar{Q}$. This also orients all of the u. In the case of a ≥ 2-arrangement this orientation determines h_u. Orientations for u and v with $u + v = V$ determine (together with the orientation of V) an orientation of $u \cap v$. Depending on whether this agrees with the orientation of $u \cap v$ defined by $(b_i^{u\cap v})_i$ or not, we set $\varepsilon_{u,v} = 1$ or $\varepsilon_{u,v} = -1$. For complex arrangements, every $u \in \bar{Q}$ has a canonical orientation, and all of the $\varepsilon_{u,v}$ will equal 1. We define these numbers more formally in the form in which we will use them.

2.2.2 Definition. Let $\mathcal{A}$ be a real arrangement and functions f^u for all $u \in \bar{Q}$ chosen as in Definition 2.1.29. Let $\bar{f}^u$ be the function defined in (2.32). For $u, v \in \bar{Q}$ with $u + v = V$ and $u \cap v \not\subset H$ we define $\varepsilon_{u,v} \in \{+1, -1\}$ by

$$\begin{aligned} &\bar{f}^u_*([\mathbb{R}P^{d(u)}, \mathbb{R}P^{d(u)-1}]) \bullet \bar{f}^v_*([\mathbb{R}P^{d(v)}, \mathbb{R}P^{d(v)-1}]) = \\ &\qquad = \varepsilon_{u,v} \bar{f}^{u\cap v}_*([\mathbb{R}P^{d(u\cap v)}, \mathbb{R}P^{d(u\cap v)-1}]), \end{aligned} \tag{2.49}$$

where the intersection product is defined by dualizing

$$H^*(PV \setminus PH, PV \setminus (PH \cup Pu)) \otimes H^*(PV \setminus PH, PV \setminus (PH \cup Pv)) \xrightarrow{\smile} H^*(PV \setminus PH, PV \setminus (PH \cup P(u \cap v))).$$

2.2.3 Theorem. *Let $\mathcal{A}$ be a real ≥ 2–arrangement. For all $u, v \in \bar{Q}$ and all $c \in H_r(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])$, $d \in H_s(\Delta[v,V], \Delta[v,V) \cup \Delta(v,V])$ we have*

$$h_u(c) \bullet h_v(d) = \begin{cases} (-1)^{r(n-d(u))} \varepsilon_{u,v} h_{u \wedge v}(c \hat{\times} d), & \text{if } u \wedge v \in \bar{Q} \\ & \text{and } d(u) + d(v) = d(u \wedge v) + n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.50}$$

The proof of this theorem will be completed at the end of this section, p. 57. This theorem has first been proved in [dLS01] (for central linear arrangements only) and in [DGM00], where it is stated in a somewhat different form.

2.2.4 Remark. If a complex arrangement is regarded as a real arrangement of double dimension, the bases can be chosen derived from complex bases, in which case all of the $\varepsilon_{u,v}$ still equal 1. For a general real arrangement this cannot always be achieved. The dependence of the cup product in the cohomology of a linear arrangement on the numbers $\varepsilon_{u,v}$ has first been shown in [Zie93], where it is used to construct two real arrangements with equalling intersection posets and dimension functions, one of them being a complex arrangement regarded as a real arrangement, with non-isomorphic cohomology rings.

2.2.5 Remark. Continuing Remark 2.1.32, the complex case of Theorem 2.2.3 can be seen as a special case of Theorem 2.2.1, since all of the isomorphisms in that remark are induced by inclusions and therefore respect the product $\hat{\times}$.

Graded formulas

We first prove graded versions of the theorems stated in the preceding section. These follow more or less for free from the algebraic machinery set up in Chapter 1. They state that the equations from the theorems we aim to prove hold at least up to error terms in higher degrees of the direct sum decompositions of the homology groups of the arrangements. They are therefore typical for the kind of result obtainable by spectral sequence arguments.

The graded versions will be an import part of the proofs of the exact versions. They hold in greater generality and in particular are true for arbitrary real arrangements, for which the exact versions fail. Our proof of the exact version for complex projective arrangements will need further more geometric arguments.

2.2.6 Proposition. *Let $\mathbb{K} = \mathbb{C}$ or let $\mathbb{K} = \mathbb{R}$ and coefficients be in $\mathbb{Z}_2$. For all $k + l \geq n$ and $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$ we have*

$$h_k(c) \bullet h_l(d) - h_{k+l-n}(c \hat{\times} d) \in \bigoplus_{i > k+l-n} \operatorname{im} h_i. \tag{2.51}$$

Proof. We want to apply Proposition 1.3.20. As the map ζ there we take the map $H(P(F^Q))\otimes_Q B(Q) \to S(P(D^{\mathcal{A}}))$ obtained as the composition of (2.19) and (2.14). We set $m := \dim_{\mathbb{R}} \mathbb{K}$. The isomorphism $\bigoplus_k H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-mk] \cong H(H(P(D^{\mathcal{A}}))\otimes_Q B(Q)\otimes_Q K^p)$ obtained by composing (2.28) with the first arrow from (1.8) is given explicitly by

$$\begin{aligned} C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) &\to H_{mk}(P(D^{\mathcal{A}})) \otimes_Q B(Q)_r \otimes_Q K^p \\ \langle q_0, \dots, q_r \rangle &\mapsto e_k^{q_0} \otimes (q_0 \leftarrow q_0 \leftarrow \cdots \leftarrow q_r \leftarrow \top) \otimes 1 \end{aligned}$$

with $1 \in K^p(\top) = R$ and $e_k^q \in H_{mk}(Pq)$ the canonical generator. By Proposition 2.1.25 the map h_k agrees with the composition of this isomorphism and the map ϕ_{mk} from Proposition 1.3.20. We note that the dimension n there equals mn in our current notation. To see that the product in that proposition agrees with $\hat{\times}$ under the above isomorphism we only have to note that $e_k^p \bullet e_l^q = e_{k+l-n}^{p\wedge q}$ for $k + l \geq n$. □

2.2.7 Proposition. *Let $u, v \in \bar{Q}$, $c \in H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])$, $d \in H_*(\Delta[v,V], \Delta[v,V) \cup \Delta(v,V])$. If $u \wedge v \in \bar{Q}$, then*

$$h_u(c) \bullet h_v(d) \in \bigoplus_{w \geq u\wedge v} \operatorname{im} h_w. \tag{2.52}$$

If additionally $d(u) + d(v) = d(u \wedge v) + n$, then

$$h_u(c) \bullet h_v(d) - (-1)^{|c|(n-d(u))}\varepsilon_{u,v} h_{u\wedge v}(c \mathbin{\hat{\times}} d) \in \bigoplus_{w > u\wedge v} \operatorname{im} h_w \tag{2.53}$$

for $\mathbb{K} = \mathbb{R}$ or

$$h_u(c) \bullet h_v(d) - h_{u\wedge v}(c \mathbin{\hat{\times}} d) \in \bigoplus_{w > u\wedge v} \operatorname{im} h_w \tag{2.54}$$

for $\mathbb{K} = \mathbb{C}$. If $u \cap v \subset I$, then

$$h_u(c) \bullet h_v(d) \in \bigoplus_{w\in \bar{Q}_{(d(u)+d(v)-n,n]}} \operatorname{im} h_w. \tag{2.55}$$

Proof. We treat the real case only, as the easier case of $\mathbb{K} = \mathbb{C}$ can be proved in the same way or derived from the case $\mathbb{K} = \mathbb{R}$.

Considering the arrangement $\{q \in Q\colon q \geq u \text{ or } q \geq v\}$, the intersection poset of which can be considered as a subset of the interval $[u \wedge v, V]$ in Q, we obtain (2.52) by naturality of the intersection product with respect to inlusion maps.

The rest of the proof proceeds as the preceding one, using a relative version of Proposition 1.3.20. We take up the notation from the proof of Proposition 2.1.29. The isomorphism

$$\bigoplus_{u\in\bar{Q}} H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])[-d(u)] \cong H(H(A(D^{\mathcal{A}})) \otimes_Q B(Q) \otimes_Q K^p)$$

is induced by

$$C_r(\Delta[u,V],\Delta[u,V)\cup\Delta(u,V]) \to H_{d(u)}(A(D^{\mathcal{A}}))\otimes_Q B(Q)_r\otimes_Q K^p$$
$$\langle q_0,\dots,q_r\rangle \mapsto e^u\otimes(u\leftarrow q_0\leftarrow\ldots\leftarrow q_r\leftarrow V)\otimes 1.$$

with $e^u := \bar{f}^u_*([\mathbb{R}P^{d(u)},\mathbb{R}P^{d(u)-1}]) \in H_{d(u)}(Pu, P(u\cap I))$, $u\in\bar{Q}$. Under this isomorphism the map ϕ_k from Proposition 1.3.20 corresponds to $\sum_{u\in\bar{Q}_{\{k\}}} h_u$.

For $d(u)+d(v) = d(u\wedge v)+n$ we have $e^u\bullet e^v = \varepsilon_{u,v}e^{u\wedge v}$ by definition of $\varepsilon_{u,v}$ and hence

$$h_u(c)\bullet h_v(d) - (-1)^{|c|(n-d(u))}\varepsilon_{u,v}h_{u\wedge v}(c\hat{\times} d) \in \bigoplus_{w\in\bar{Q}_{(k+l-n,n]}} \operatorname{im} h_w.$$

Since

$$\bigoplus_{w\in\bar{Q}_{(k+l-n,n]}} \operatorname{im} h_w \cap \bigoplus_{w\geq u\wedge v} \operatorname{im} h_w = \bigoplus_{w>u\wedge v} \operatorname{im} h_w,$$

this proves (2.53).

For $u\cap v\subset I$, $e^u\bullet e^v = 0$ by necessity, since $H(A(u\wedge v)) = 0$, and the above argument yields (2.55). □

2.2.8 Remark. Proposition 2.2.7 is proved in [dLS01] as Theorem 7.5. While there it is more of an afterthought to the exact version, it appears here as a very natural result in its own right and a possible basis to a proof of the exact version.

2.2.9 Remark. If n is odd, then $\mathbb{R}P^n$ is orientable and Proposition 1.3.20 is applicable to $H_*(PV,\bigcup P\mathcal{A})$ also for $\mathbb{K}=\mathbb{R}$, $R=\mathbb{Z}$, since in the proof of Proposition 2.1.20 the needed ZŽ–map is constructed. In the product $H(Pu)\otimes H(Pv)\xrightarrow{\bullet} H(P(u\wedge v))$ the product of two generators is a generator whenever possible and to determine the sign of the product of two generators of $\mathbb{Z}$-summands, orientation information is needed as in the affine case.

Inductive proofs of product formulas

We now reduce Theorem 2.2.1 and Theorem 2.2.3 to the cases where they state that products be zero because of their degrees. This is done using the graded versions presented in the preceding section. The inductive step is made possible by the results on intersections with a hyperplane presented in Section 2.1.

2.2.10 Proposition. *If it is true for all complex arrangements $\mathcal{A}$ of all dimensions that $h_k(c)\bullet h_l(d) = 0$ for $c\in H_*(\Delta Q_{[k,n]},\Delta Q_{[k,n)})$, $d\in H_*(\Delta Q_{[l,n]},\Delta Q_{[l,n)})$, whenever $k+l<n$, then Theorem 2.2.1 is true.*

Proof. We have to show that $h_k(c)\bullet h_l(d) = h_{k+l-n}(c\hat{\times} d)$, where we use the convention that $h_i = 0$ for $i<0$.

We proceed by induction on the dimension of V.

For $k+l<n$ the conjecture is covered by the assumption. For $k+l \geq n$ and $\dim V = 1$ it is covered by Proposition 2.2.6 (and trivial anyway). For $k+l \geq n$ and $\dim V > 1$ we choose a hyperplane in general position with respect to the arrangement and adopt the notation of Section 2.1. By induction and Proposition 2.1.35 (for $k+l>n$) or the assumption (for $k+l=n$)

$$\begin{aligned} i_!(h_k(c) \bullet h_l(d)) = i_!(h_k(c)) \bullet i_!(h_l(d)) = h^H_{k-1}(\eta_*(c)) \bullet h^H_{l-1}(\eta_*(d)) = \\ = h^H_{k+l-n-1}(\eta_*(c) \hat{\times} \eta_*(d)) = h^H_{k+l-n-1}(\eta_*(c \hat{\times} d)) = i_!(h_{k+l-n}(c \hat{\times} d)). \end{aligned}$$

Again by Proposition 2.1.35, this implies $h_k(c) \bullet h_l(d) - h_{k+l-n}(c \hat{\times} d) \in \ker i_! = \operatorname{im} h_0$. But $h_k(c) \bullet h_l(d) - h_{k+l-n}(c \hat{\times} d) \in \bigoplus_{i>k+l-n} \operatorname{im} h_i$ by Proposition 2.2.6. Therefore $h_k(c) \bullet h_l(d) - h_{k+l-n}(c \hat{\times} d) = 0$. □

2.2.11 Proposition. *If it is true for all real ≥ 2-arrangements $\mathcal{A}$ of all dimension and all $u,v \in \bar{Q}$ that $h_u(c) \bullet h_v(d) = 0$ for $c \in H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])$ and $d \in H_*(\Delta[v,V], \Delta[v,V) \cup \Delta(v,V])$ whenever $d(u)+d(v)<n$, then Theorem 2.2.3 is true.*

Proof. The proof proceeds by induction on the dimension of V, parallel to the preceding one.

Let $u,v \in \bar{Q}$, $d(u)+d(v) \geq n$.

If $\dim V = 1$, then Theorem 2.2.3 holds trivially for V. We assume $\dim V > 1$ and let $H \subset V$ be a hyperplane in general position with respect to $\mathcal{A}$ and I. For simplicity, we also assume $d(u), d(v) > 0$, because otherwise $u = V$ or $v = V$ and these cases are easily dealt with directly.

If $u \cap v \subset I$, then $\eta(u) \cap \eta(v) \subset I \cap H$. If $u \wedge v \in \bar{Q}$ and $d(u)+d(v)-d(u \wedge v) < n$ then $\eta(u) \cap \eta(v) \subset I \cap H$ or $d(\eta(u)) + d(\eta(v)) - d(\eta(u) \wedge \eta(v)) < n-1$. In these cases

$$i_!(h_u(c) \bullet h_v(d)) = i_!(h_u(c)) \bullet i_!(h_v(d)) = h^H_{\eta(u)}(\eta_*(c)) \bullet h^H_{\eta(v)}(\eta_*(d)) = 0$$

and hence $h_u(c) \bullet h_v(d) \in \bigoplus_{w \in \bar{Q}_{\{0\}}} h_w$ by induction and Proposition 2.1.36, and $h_u(c) \bullet h_v(d) \in \bigoplus_{w \in \bar{Q}_{(0,n]}} h_w$ by Proposition 2.2.7. Therefore $h_u(c) \bullet h_v(d) = 0$.

Now let $u+v=V$, $u \cap v \not\subset I$. If $d(u)+d(v)=n$, then

$$\begin{aligned} i_!\big(h_u(c) \bullet h_v(d) - (-1)^{|c|(n-d(u))} \varepsilon_{u,v} h_{u \wedge v}(c \hat{\times} d)\big) = \\ = h^H_{\eta(u)}(\eta_*(c)) \bullet h^H_{\eta(v)}(\eta_*(d)) = 0 \end{aligned}$$

by induction and Proposition 2.1.36. If $d(u)+d(v)>n$, we make sure that $h^H_{\eta(u)}$, $h^H_{\eta(v)}$, $h^H_{\eta(u \wedge v)}$ are defined with orientations compatible with those underlying h_u, h_v, $h_{u \wedge v}$ as in Proposition 2.1.36. Then

$$\begin{aligned} &(\bar{f}^{\eta(u)}_H)_*([\mathbb{R}P^{d(\eta(u))}, \mathbb{R}P^{d(\eta(u))-1}]) \bullet (\bar{f}^{\eta(v)}_H)_*([\mathbb{R}P^{d(\eta(v))}, \mathbb{R}P^{d(\eta(v))-1}] \\ &= i_!\big(\bar{f}^u_*([\mathbb{R}P^{d(u)}, \mathbb{R}P^{d(u)-1}]) \bullet \bar{f}^v_*([\mathbb{R}P^{d(v)}, \mathbb{R}P^{d(v)-1}])\big) \\ &= i_!(\varepsilon_{u,v}(\bar{f}^{u \wedge v}_H)_*([\mathbb{R}P^{d(u \wedge v)}, \mathbb{R}P^{d(u \wedge v)-1}]) \\ &= \varepsilon_{u,v}(\bar{f}^{\eta(u) \wedge \eta(v)}_H)_*([\mathbb{R}P^{d(\eta(u) \wedge \eta(v))}, \mathbb{R}P^{d(\eta(u) \wedge \eta(v))-1}]) \end{aligned}$$

and hence $\varepsilon^H_{\eta(u),\eta(v)} = \varepsilon_{u,v}$. Again, this yields

$$i_!\big(h_u(c) \bullet h_v(d) - (-1)^{|c|(n-d(u))}\varepsilon_{u,v} h_{u\wedge v}(c \hat{\times} d)\big) = \\ h^H_{\eta(u)}(\eta_*(c)) \bullet h^H_{\eta(v)}(\eta_*(d)) - (-1)^{|c|(n-1-d(\eta(u)))}\varepsilon^H_{\eta(u),\eta(v)} h^H_{\eta(u)\wedge\eta(v)}(c \hat{\times} d) \\ = 0.$$

In both cases, we combine this with Proposition 2.1.36 and Proposition 2.2.7 to get $h_u(c) \bullet h_v(d) - (-1)^{|c|(n-d(u))}\varepsilon_{u,v} h_{u\wedge v}(c \hat{\times} d) \in \bigoplus_{w\in\bar{Q}_{\{0\}}} h_w \cap \bigoplus_{w>u\wedge v} h_w = 0$. □

Vanishing for affine arrangements

For affine ≥ 2-arrangements the vanishing of intersection products in the cases required by Proposition 2.2.11 is easily proved, since it is indeed possible to find chains representing the involved hology classes that do not intersect geometrically.

2.2.12 Proposition. *Let $\mathcal{A}$ be a ≥ 2–arrangement, $u, v \in \bar{Q}$, $d(u) + d(v) < n$, $c \in H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])$, $d \in H_*(\Delta[u,V], \Delta[u,V) \cup \Delta(u,V])$. Then $h_u(c) \bullet h_v(d) = 0$.*

Proof. Let Λ be a linear functional on V with $\ker \Lambda = H$.

There is a linear functional $F \neq 0$ on V with $u \subset \ker F$ and $v \cap H \subset \ker F$. We choose b^u, x^u to define f^u and $\tilde{b}^v$, $\tilde{x}^v$ to define $\tilde{f}^v$ as in Definition 2.1.29. This can be done in such a way that $F(x^u_w) \leq 0$ for all $w \geq u$, $F(x^v_w) \geq 0$ for all $w \geq v$, $F(x^u_V) < 0$, $F(x^v_V) > 0$ and $\Lambda(x^u_u) = \Lambda(\tilde{x}^v_v) = 1$.

We set

$$X^+ := \big\{[z] \in PV \colon \Lambda(z) \neq 0, \Lambda(z)^{-1}F(z) \geq 0\big\} \cup PH,$$
$$X^- := \big\{[z] \in PV \colon \Lambda(z) \neq 0, \Lambda(z)^{-1}F(z) \leq 0\big\} \cup PH.$$

Then

$$\operatorname{im} f^u \subset X^-, \qquad (f^u)^{-1}[X^+ \cap X^-] \subset \mathbb{K}P^k \times \Delta[u,V) \cup \mathbb{K}P^{k-1} \times \Delta[u,V],$$
$$\operatorname{im} \tilde{f}^v \subset X^+, \qquad (\tilde{f}^v)^{-1}[X^+ \cap X^-] \subset \mathbb{K}P^l \times \Delta[v,V) \cup \mathbb{K}P^{l-1} \times \Delta[v,V].$$

Therefore $\operatorname{im} f^u \cap \operatorname{im} \tilde{f}^v \subset \bigcup P\mathcal{A} \cup PH$ and $h_u(c) \bullet h_v(d) = 0$. □

Proof of Theorem 2.2.3. The theorem follows directly from Proposition 2.2.12 and Proposition 2.2.11. □

2.3 Products in projective arrangements

In this section we will complete the proof of Theorem 2.2.1, the product formula for complex projective arrangements. We have already reduced this in Proposition 2.2.10 to the case of $h_k(c) \bullet h_l(d)$ with $k + l < n$, which we have to show to be zero. The corresponding fact for affine arrangements could be proved by representing the homology classes by chains which do not intersect geometrically. A proof along these lines seems not to be available for projective arrangements. We will first consider an example of a real projective arrangement where an intersection product of this kind is indeed not zero. Doing this we will also try to gain some intuition on why in the projective case the intersection of the chains should not make a homological contribution, even if existing geometrically. We will then develop the techniques necessary to transform this intuition into a proof.

An example of real projective arrangements

We will see how the product formula of Theorem 2.2.1 fails for real projective arrangements and sketch the difference between real and complex arrangements that will allow us to prove the formula for complex projective arrangements.

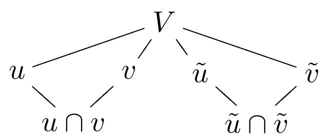
Let $k, l \geq 0$, $n := k + l + 1$. We consider the following subspaces of $\mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^2$.

$$u := \mathbb{R}^k \times \{0\} \times (\mathbb{R} \cdot (0,1)), \qquad \tilde{u} := \{0\} \times \mathbb{R}^l \times (\mathbb{R} \cdot (1,1)),$$
$$v := \mathbb{R}^k \times \{0\} \times (\mathbb{R} \cdot (4,1)), \qquad \tilde{v} := \{0\} \times \mathbb{R}^l \times (\mathbb{R} \cdot (5,1)),$$

In the arrangement $\check{\mathcal{A}} := \{u, v, \tilde{u}, \tilde{v}\}$ we will find classes $c \in H(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$ with $h_k(c) \bullet h_l(d) \neq 0$, although $k + l < n$.

The combinatorial data of $\check{\mathcal{A}}$ are given by the intersection poset

$\check{Q}_{[0,n]} =$



with $u \cap v$ and $\tilde{u} \cap \tilde{v}$ only present for $k > 0$ and $l > 0$ respectively, and the dimensions $d(u) = d(v) = k$, $d(u \cap v) = k - 1$, $d(\tilde{u}) = d(\tilde{v}) = l$, $d(\tilde{u} \cap \tilde{v}) = l - 1$, $d(V) = n$. In case of $k = l$, we can, if we want to, avoid the intersections $u \cap v$ and $\tilde{u} \cap \tilde{v}$ by a small change of u and $\tilde{u}$ without substantially affecting the calculations below. This shows that, in contrast to the case of affine arrangements, a simple condition on the occuring codimensions will not be enough for the product formula to extend from complex to real arrangements.

To simplify the pictures below and to have the notation parallel that of Section 2.3, we consider $\check{A}$ to be the union of the two arrangements $\mathcal{A} := \{u, v\}$ and $\tilde{A} := \{\tilde{u}, \tilde{v}\}$. The arrangement $\tilde{\mathcal{A}}$ is in general position with respect to $\mathcal{A}$ as in Definition 2.3.4.

Denoting the intersection posets of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ by Q and $\tilde{Q}$ respectively, we have $H_1(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, generated by $c := [\langle u, V\rangle + \langle v, V\rangle]$, and $H_1(\Delta\tilde{Q}_{[l,n]}, \Delta\tilde{Q}_{[l,n)}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, generated by $d := [\langle \tilde{u}, V\rangle + \langle \tilde{v}, V\rangle]$. For the definition of

$$f^k \colon \mathbb{R}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \to (V, \bigcup P\mathcal{A}),$$
$$\tilde{f}^l \colon \mathbb{R}P^l \times (\Delta\tilde{Q}_{[k,n]}, \Delta\tilde{Q}_{[k,n)}) \to (V, \bigcup P\tilde{\mathcal{A}}),$$

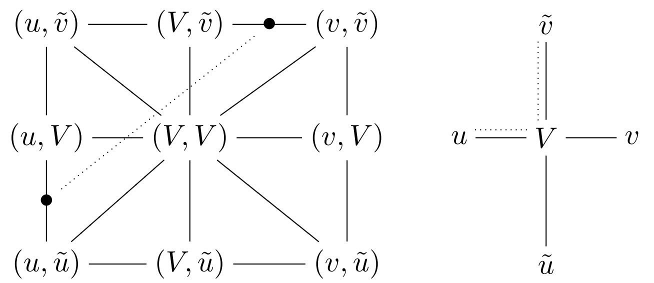
and hence of

$$h_k \colon H_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}; \mathbb{Z}_2) \to H_{r+k}(V, \bigcup P\mathcal{A}; \mathbb{Z}_2),$$
$$\tilde{h}_l \colon H_r(\Delta\tilde{Q}_{[k,n]}, \Delta\tilde{Q}_{[k,n)}; \mathbb{Z}_2) \to H_{r+l}(V, \bigcup P\tilde{\mathcal{A}}; \mathbb{Z}_2)$$

we set, with $(e_0, \ldots, e_n)$ the standard basis of $V = \mathbb{R}^{n+1}$,

$$x_j^u := \begin{cases} e_j, & j < k, \\ e_{k+l+1}, & j = k, \end{cases} \qquad \tilde{x}_j^{\tilde{u}} := \begin{cases} e_{k+j}, & j < l, \\ e_{k+l} + e_{k+l+1}, & j = l, \end{cases}$$
$$x_j^v := \begin{cases} e_j, & j < k, \\ 4e_{k+l} + e_{k+l+1}, & j = k, \end{cases} \qquad \tilde{x}_j^{\tilde{v}} := \begin{cases} e_{k+j}, & j < l, \\ 5e_{k+l} + e_{k+l+1}, & j = l, \end{cases}$$
$$x_j^V := \begin{cases} e_j, & j < k, \\ 2e_{k+l} + e_{k+l+1}, & j = k, \end{cases} \qquad \tilde{x}_j^V := \begin{cases} e_{k+j}, & j < l, \\ 3e_{k+l} + e_{k+l+1}, & j = l. \end{cases}$$

To determine $h_k(c) \bullet h_l(d)$, we first have a look at the geometric intersection $S := f^k\left[\mathbb{R}P^k \times \Delta Q_{[k,n]}\right] \cap f^l\left[\mathbb{R}P^l \times \Delta\tilde{Q}_{[l,n]}\right]$. For $x \in \Delta(Q_{[k,n]})$, $y \in \Delta(\tilde{Q}_{[l,n]})$, the intersection $f\left[\mathbb{R}P^k \times \{x\}\right] \cap f\left[\mathbb{R}P^l \times \{y\}\right]$ is either empty or consists of a single point. The left of the following two pictures shows the two dimensional simplicial complex $\Delta Q_{[k,n]} \times \Delta\tilde{Q}_{[l,n]} = \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})$.



(2.56)

The dotted line depicts the set $\bar{S}$ of those points (x, y) for which the intersection is nonempty. S is a connected 1–dimensional manifold with boundary, and we can see from the picture that one boundary point lies in $u \cap V = u$ and the other one in $V \cap \tilde{v} = \tilde{v}$. A closer look at S, which is the intersection of two manifolds that meet transversely, shows that indeed $h_k(c) \bullet \tilde{h}_l(d) = \check{h}_0([\langle u, V\rangle + \langle \tilde{v}, V\rangle])$. $[\langle u, V\rangle + \langle \tilde{v}, V\rangle]$ is a generator of $H_1(\Delta\check{Q}_{[0,n]}, \Delta\check{Q}_{[0,n)}; \mathbb{Z}_2)$, therefore $h_k(c) \bullet \tilde{h}_l(d) \neq 0$.

We equip $Q \times \tilde{Q}$ with a dimension function $d(p, q) := d(p) + d(q)$. The map $Q \times \tilde{Q} \to \check{Q}$, $(p, q) \mapsto p \cap q$, sends $((Q \times \tilde{Q})_{[n,2n]}, (Q \times \tilde{Q})_{[n,2n)})$ to $(\check{Q}_{[0,n]}, \check{Q}_{[0,n)})$.

In the picture, the border of the square is $\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)}$ and the four vertices at the corners are $\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,n)}$. Under the composition of maps

$$\begin{array}{c} \big(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}) \setminus \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,n)}, \\ \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)} \setminus \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,n)}\big) \\ \downarrow \\ (\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[n,2n]}, \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[n,2n)}) \\ \downarrow \\ (\Delta\check{Q}_{[0,n]}, \Delta\check{Q}_{[0,n)}), \end{array} \tag{2.57}$$

the first being a deformation retraction and the second given by inclusion, in our example $(\bar{S}, \mathfrak{d}\bar{S})$ is mapped to the dotted line in the picture on the right. This set carries the relative cycle $\langle u, V\rangle + \langle \tilde{v}, V\rangle$ representing $h_k(c) \bullet \tilde{h}_l(d)$. We will see in Section 2.3 that this is not just a coincidence.

When considering complex arrangements we will see that in the above situation we gain one dimension compared to real arrangements, and $\bar{S}$ will miss the cone with top the vertix (V,V) and base $\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,n)}$. The map of $(\bar{S}, \mathfrak{d}\bar{S})$ to $(\Delta\check{Q}_{[0,n]}, \Delta\check{Q}_{[0,n)})$ will therefore miss the vertix V and be homotopic to a map with image in $\Delta\check{Q}_{[0,n)}$. This is the idea behind the proof of Proposition 2.3.14, although it will be technically a bit different.

The product of two arrangements and general position

The intersection of two sets can be identified with the intersection of their cartesian product with a diagonal. Similarly the intersection product of two homology classes equals the image of their cross product under the image of the transfer map associated with the diagonal map. We therefore study the products of two arrangements.

Already in the real example we have discussed, it was useful to assume the homology classes of which the product was to be determined to be carried by different arrangements. So we now assume to be given a second arrangement $\tilde{A}$ in V with intersection poset $\tilde{Q}$. For the first part of this section the arrangement could be in a vector space different from V, but we will have no use for this generality later on.

We equip the poset $Q \times \tilde{Q}$ with a dimension function d by $d(u,v) := d(u) + d(v)$. The counterpart of h for $\tilde{A}$ will be denoted by $\tilde{h}$ and so on.

As noted above, we will be interested in cross products.

2.3.1 Definition and Proposition. *Any choice of* $(y_i^{u,v})_{i=0,\ldots,k}$, $(z_i^{u,v})_{i=0,\ldots,l}$ *for* $(u,v) \in Q_{[k,n]} \times \tilde{Q}_{[l,n]}$ *with* $y_i^{u,v} \in u$, $z_i^{u,v} \in v$ *and such that for all* $(u_0, v_0) <$

$\cdots < (u_m, v_m)$ *and* $\lambda \in \Delta^m$ *the system* $(\sum_j \lambda_j y_i^{u_j,v_j})_i$ *as well as the system* $(\sum_j \lambda_j z_i^{u_j,v_j})_i$ *is linearly independent, yields a map*

$$g\colon \mathbb{C}P^k \times \mathbb{C}P^l \times \big(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}), \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)}\big) \to (PV, \bigcup P\mathcal{A}) \times (PV, \bigcup P\tilde{\mathcal{A}})$$
$$\left([\mu_0 : \cdots : \mu_k], [\nu_0 : \cdots : \nu_l], \sum_j \lambda_j (u_j, v_j)\right) \mapsto \left(\left[\sum_{i,j} \lambda_j \mu_i y_i^{u_j,v_j}\right], \left[\sum_{i,j} \lambda_j \nu_i z_i^{u_j,v_j}\right]\right).$$

As in Definition 2.1.25, any two such maps are homotopic for $\mathbb{K} = \mathbb{C}$. □

2.3.2 Proposition. *Let* $c \in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$, $d \in H_*(\Delta\tilde{Q}_{[l,n]}, \Delta\tilde{Q}_{[l,n)})$, *and* $\mathbb{K} = \mathbb{C}$. *Then* $h_k(c) \times \tilde{h}_l(d) = g_*\left([\mathbb{C}P^k] \times [\mathbb{C}P^l] \times (c \times d)\right)$.

Proof. Since $\mathbb{K} = \mathbb{C}$, the homomorphisms h_k, $\tilde{h}_l$ and g_* do not depend on the choices made in defining them. For the choice $y_i^{u,v} = x_i^u$, $z_i^{u,v} = \tilde{x}_i^v$, we just get the map $f^k \times \tilde{f}^l$ up to identification of

$$\mathbb{C}P^k \times \mathbb{C}P^l \times \left(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}), \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)}\right)$$

with $\mathbb{C}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \times \mathbb{C}P^l \times (\Delta\tilde{Q}_{[l,n]}, \Delta\tilde{Q}_{[l,n)})$. Again since $\mathbb{K} = \mathbb{C}$, no sign is introduced by the interchange of factors made in this identification. □

As noted after discussing the real example, it will be important to control the codimension of a set corresponding to the dotted line in (2.56). We will now work towards this and start with an algebraic lemma.

2.3.3 Lemma. *Let* $\mathbb{K} = \mathbb{C}$. *Let* u, v *be subspaces of* V *in general position with respect to each other,* $\dim u = r \geq k+1$, $\dim v = s \geq l+1$, $\dim V = n+1$, $k + l < n$. *Let* O *be the open subspace of the affine space* $u^{k+1} \times v^{l+1}$ *defined by*

$$O := \left\{(y_0, \ldots, y_k, z_0, \ldots, z_l) \colon \dim(\operatorname{span}\{y_i\}) = k+1,\ \dim(\operatorname{span}\{z_i\}) = l+1\right\}$$

and algebraic subsets $\cdots \subset S_1 \subset S_0 \subset O$ *defined by*

$$S_m := \{(y_0, \ldots, y_k, z_0, \ldots, z_l) \colon \dim(\operatorname{span}(\{y_i\} \cup \{z_j\})) < k + l + 2 - m\}$$

Then $S_m \setminus S_{m+1}$ *is a complex submanifold of codimension* $(1+m)(n-k-l+m)$.

Proof. We consider $(y_0, \ldots, y_k, z_0, \ldots, z_l) \in S_m \setminus S_{m+1}$. This implies $u + v = V$. We set $Y := \operatorname{span}\{y_i\}$, $t := n - k - s + \dim(Y \cap v)$, and choose a basis $(e_0, \ldots, e_n)$ of V such that $\operatorname{span}\{e_0, \ldots, e_{r-1}\} = u$, $\operatorname{span}\{e_{n-s+1}, \ldots, e_n\} = v$, $\operatorname{span}\{e_t, \ldots, e_{k+1}\} = Y$. Let A be the $(n+1) \times (k+l+2)$–matrix with columns $(y_0, \ldots, y_k, z_0, \ldots, z_l)$ expressed using this basis. Elements of O are represented by matrices $A' = (a'_{ij})$ with $a'_{ij} = 0$ for $r \leq i \leq n$, $0 \leq j \leq k$ and for $0 \leq i \leq n - s$, $k+1 \leq j \leq k+l+1$ such that the first $k+1$ and the last $l+1$ columns are linearly independent, and $A = (a_{ij})$ has the additional property that the first t rows are zero.

There are sets I and J with $\{t, \ldots, k+t\} \subset I \subset \{t, \ldots, n\}$, $\{0, \ldots, k\} \subset J \subset \{0, \ldots, k+l+1\}$ and $|I| = |J| = k + l + 1 - m$ such that the matrix $B :=$

$(a_{ij})_{i\,\in\,I, j\,\in\,J}$ is regular. Similarly, there exist I', J' with $I' \subset \{t, \ldots, n\} \setminus M$, $\{k+1, \ldots, k+l+1\} \subset J' \subset \{0, \ldots, k+l+1\}$ and $|I|' = |J|' = k+l+1-m$ such that the matrix $C := (a_{ij})_{i\,\in\,I', j\,\in\,J'}$ is regular.

Let $U \subset O$ be a neighbourhood of A such that for every $A' = (a'_{ij}) \in U$ the matrices $(a'_{ij})_{i\,\in\,I, j\,\in\,J}$ and $(a'_{ij})_{i\,\in\,I', j\,\in\,J'}$ are regular. Then an $A' \in U$ is in S_m if and only if the equations

$$f_{i_0 j_0}(A') := \det(a'_{ij})_{\substack{i \in I \cup \{i_0\} \\ j \in J \cup \{j_0\}}} = 0 \qquad \text{for all } i_0 \in I_0 := \{n+1-s, \ldots, n\} \setminus I,$$
$$j_0 \in J_0 := \{0, \ldots, k+l+1\} \setminus J$$

and

$$g_{i_0 j_0}(A') := \det(a'_{ij})_{\substack{i \in I' \cup \{i_0\} \\ j \in J' \cup \{j_0\}}} = 0 \qquad \text{for all } i_0 \in \{0, \ldots, t-1\},$$
$$j_0 \in J'_0 := \{0, \ldots, k+l+1\} \setminus J'$$

hold. To see this, assume $A' \notin S_m$, i.e. $\operatorname{rk} A' > k+l+1-m$. If the rank of the matrix A' with the first t rows deleted is greater than $k+l+1-m$, one of the functions $f_{i_0 j_0}$ becomes non-zero, otherwise one of the functions $g_{i_0 j_0}$.

Finally we compute for $(i_1, j_1) \in I_0 \times J_0$

$$\left| \frac{\partial f_{i_0 j_0}(A)}{\partial a_{i_1 j_1}} \right| = \begin{cases} |\det B|, & (i_0, j_0) = (i_1, j_1), \\ 0, & (i_0, j_0) \neq (i_1, j_1), \end{cases} \qquad (i_0, j_0) \in I_0 \times J_0,$$
$$\frac{\partial g_{i_0 j_0}(A)}{\partial a_{i_1 j_1}} = 0, \qquad (i_0, j_0) \in \{0, \ldots, t-1\} \times J'_0$$

and for $(i_1, j_1) \in \{0, \ldots, t-1\} \times J'_0$

$$\frac{\partial f_{i_0 j_0}(A)}{\partial a_{i_1 j_1}} = 0, \qquad (i_0, j_0) \in I_0 \times J_0,$$
$$\left| \frac{\partial g_{i_0 j_0}(A)}{\partial a_{i_1 j_1}} \right| = \begin{cases} |\det C|, & (i_0, j_0) = (i_1, j_1), \\ 0, & (i_0, j_0) \neq (i_1, j_1), \end{cases} \qquad (i_0, j_0) \in \{0, \ldots, t-1\} \times J'_0$$

and $|I_0 \times J_0 \cup \{0, \ldots, t-1\} \times J'_0| = (n+1-|I|)(m+1) = (n-k-l+m) \cdot (m+1)$. □

2.3.4 Definition. We say that the arrangement $\tilde{\mathcal{A}}$ is in general position with respect to the arrangement $\mathcal{A}$, if for all $u \in Q$ and $v \in \tilde{Q}$, we have $u \cap v = 0$ whenever $d(u) + d(v) < n$ and $d(u \cap v) = d(u) + d(v) - n$ otherwise.

2.3.5 Proposition. *Let $\mathbb{K} = \mathbb{C}$, $k + l < n$, $D \subset PV \times PV$ be the diagonal and $S \subset \Delta(Q_{[k,n]} \times Q_{[l,n]})$ be defined as the set of all points x such that $g\left[\mathbb{C}P^k \times \mathbb{C}P^l \times \{x\}\right] \cap D \neq \emptyset$. For a generic choice of the points $y_i^{u,v}$ and $z_i^{u,v}$ defining g, the set S intersects every open simplex of $\Delta(Q_{[k,n]} \times Q_{[l,n]})$ in an algebraic set of real codimension $2(n-k-l)$.*

Proof. In regard of Lemma 2.3.3 all that is required is that for each chain $(u_0, v_0) < \cdots < (u_t, v_t)$ the affine plane in $u_t^{k+1} \times v_t^{k+1}$ spanned by the $t+1$ points $(y^{u_0,v_0}, z^{u_0,v_0}), \ldots, (y^{u_t,v_t}, z^{u_t,v_t})$ meets the algebraic set S_0 transversely. Assuming that the affine plane spanned by the first t of these points already meets S_0 transversely, this will be fulfilled for a generic choice of $(y^{u_t,v_t}, z^{u_t,v_t}) \in u_t^{k+1} \times v_t^{k+1}$. □

Recovering the direct sum decomposition

When discussing the real example, it seemed plausible that a certain subset of the order complex of the intersection poset *should* carry the inverse image of the considered intersection product under the isomorphism $\sum_k h_k$. We now develop tools that allow to actually *prove* this kind of proposition.

More generally, given a class in $H_*(PV, \bigcup P\mathcal{A})$ we want to determine the corresponding element of $\bigoplus_k H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$. Because of Proposition 2.1.35 it will suffice to identify the part in the summand $H_*(\Delta Q_{[0,n]}, \Delta Q_{[0,n)})$. The key to this will be to not only consider the map $f^0 \colon \Delta Q_{[0,n]} \to PV$, but also a map $PV \to Q_{[0,n]}$, where the poset Q is topologized in an appropriate way yielding the *space of strata*. While we have up to this point used only the former map, in [DGM00] a description of the cohomology ring of the complement of an affine arrangement is obtained using exclusively the latter map. Here the interplay of both maps will be important.

2.3.6 Definition. Let P be a poset. We make P into a topological space by calling a set $O \subset P$ open, iff $x \in O$ implies $y \in O$ for all $y \geq x$.

2.3.7 Lemma. *Let X be a space, P a poset, $A \subset X$, $R \subset P$. If $f, g \colon (X, A) \to (P, R)$ are continuous maps with $f(x) \geq g(x)$ for all $x \in X$, then $f \simeq g$.*

Proof. The desired homotopy is given by

$$\begin{aligned} H \colon (X, A) \times I &\to (P, R) \\ (x, t) &\mapsto \begin{cases} f(x), & t < 1, \\ g(x), & t = 1. \end{cases} \end{aligned}$$

This map is continuous, since $g^{-1}[O] \subset f^{-1}[O]$ for open $O \subset X$, and therefore $H^{-1}[O] = f^{-1}[O] \times [0,1) \cup g^{-1}[O] \times \{1\} = f^{-1}[O] \times [0,1) \cup g^{-1}[O] \times I$. □

2.3.8 Lemma. *If P has a minimum or a maximum, then P is contractible.*

Proof. By the preceding lemma, the constant map to the minimum respectively the maximum is homotopic to the identity. □

2.3.9 Definition. Let X be a space equipped with a covering $\mathfrak{C}$ by closed sets and let P be the poset $P := \{\bigcap M : \emptyset \neq M \subset \mathfrak{C}, \bigcap M \neq \emptyset\}$, odered by inclusion. We define a continous map

$$\begin{aligned} s\colon X &\to P \\ x &\mapsto \min\{p \in P : x \in P\} = \bigcap\{C \in \mathfrak{C} : x \in C\}\,. \end{aligned}$$

In particular we consider the following two kinds of maps. For our arrangement $\mathcal{A}$ we consider the map $s^{\mathcal{A}}\colon PV \to Q_{[0,n]}$ corresponding to the covering $P\mathcal{A} \cup \{PV\}$ of PV. For a poset P which has unique minima in the sense that for $M \subset P$, $M \neq \emptyset$, the set $\{p \in P : p \leq q \text{ for all } q \in M\}$ is either empty or of the form $\{p : p \leq q\}$ for a $q \in P$, we consider the map $s^P\colon \Delta P \to P$ arising from the covering of ΔP by the subspaces $\Delta(\{p' : p' \leq p\})$, $p \in P$.

2.3.10 Lemma. *For a finite poset P and $R \subset P$, both satisfying the condition regarding minima of the preceding definition, the map $s^P_*\colon H_*(\Delta P, \Delta R) \xrightarrow{\cong} H_*(P, R)$ is an isomorphism.*

Proof. We may assume $R = \emptyset$, because the general case will follow by an application of the five lemma.

We consider the covering of P by the open subsets $X(p) := \{q : q \geq p\}$. These together with the inclusion maps form a P-diagram of spaces. If $\mathbb{Z}$ denotes the constant diagram, then $S(X) \otimes_{P^o} \mathbb{Z} = \sum_{p \in P} S(X(p))$ and $H(S(X) \otimes_{P^o} \mathbb{Z}) \cong H(P)$ induced by inclusion. As in Proposition 1.3.9 the diagram $S(X)$ is free. For this note that the existence of maxima follows from the existence of minima. Since s^P maps $\Delta X(p)$ to $X(p)$, it induces a map $\mathbb{Z} \otimes_{P^o} B(P^o) \to S(X)$. Regarding $\mathbb{Z}$ as a chain complex concentrated in dimension 0 this is a ZŽ–map, because $X(p)$ is acyclic for all p. The resulting isomorphism

$$H_*(\Delta P) \cong H(\mathbb{Z} \otimes_{P^o} B(P^o) \otimes_{P^o} \mathbb{Z}) \to H(S(X) \otimes_{P^o} \mathbb{Z}) \cong H(P)$$

is easily identified with s^P_*. □

2.3.11 Proposition. *The composition*

$$H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \xrightarrow{h_k} H_*(PV, \bigcup P\mathcal{A}) \xrightarrow{s^{\mathcal{A}}_*} H_*(Q_{[0,n]}, Q_{[0,n)})$$

is an isomorphism for $k = 0$ and zero for $k > 0$.

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathbb{C}P^k \times (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) & & \xrightarrow{\quad f^k \quad} & & (PV, \bigcup P\mathcal{A}) \\ \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle s^{\mathcal{A}}} \\ (\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) & \xrightarrow{s^{Q_{[k,n]}}} & (Q_{[k,n]}, Q_{[k,n)}) & \xrightarrow{i} & (Q_{[0,n]}, Q_{[0,n)}) \end{array}$$

where i is the inclusion map and π the projection onto the second factor. By construction of f^k, $f^k\left[\{x\}\times\langle u_0,\dots,u_q\rangle\right]\subset Pu_q$, that is $s^{\mathcal{A}}(f^k(x,y))\le s^Q(y)$, and by Lemma 2.3.7 this implies the homotopy commutativity of the diagram.

For $k=0$, π is a homeomorhism, i the identity, and h_0 equals f^0_* up to an isomorphism. Therefore $s^{\mathcal{A}}_*\circ h_0$ is an isomorphism, because $s_*^{Q_{[0,n]}}$ is an isomorphism by Lemma 2.3.10.

For $k>0$, $s^{\mathcal{A}}_*(h_k(c)) = s^{\mathcal{A}}_*(f^k_*([\mathbb{C}P^k]\times c)) = (i\circ s^{Q_{[k,n]}})_*(\pi_*([\mathbb{C}P^k]\times c)) = (i\circ s^{Q_{[k,n]}})_*(0)=0$. □

Since we are concerned with the vanishing of certain intersection products, we will use the following immediate corollary.

2.3.12 Corollary. *Let $0\le i\le n$, $c_i\in H_*(\Delta Q_{[k,n]},\Delta Q_{[k,n)})$, and $x=\sum_i h_i(c_i)\in H_*\left(PV,\bigcup P\mathcal{A}\right)$. If $s^{\mathcal{A}}_*(x)=0\in H_*(Q_{[0,n]},Q_{[0,n)})$, then $c_0=0$.* □

Vanishing for projective arrangements

We are now ready to prove the last step in the proof of the product formula for complex projective arrangements, namely the following proposition.

2.3.13 Proposition. *Let $\mathbb{K}=\mathbb{C}$ and $k+l<n$, $c\in H_*(\Delta Q_{[k,n]},\Delta Q_{[k,n)})$, $d\in H_*(\Delta Q_{[l,n]},\Delta Q_{[l,n)})$. Then*

$$h_k(c)\bullet h_l(d)=0.$$

We will assume $\mathbb{K}=\mathbb{C}$ from now on. We will prove the proposition in three steps.

We would like to have the classes $h_k(c)$ and $h_l(d)$ represented by chains as much as possible in general position with respect to each other. To this end we consider an arrangement that is the union of two arrangements $\mathcal{A}$ and $\tilde{\mathcal{A}}$ with intersection posets Q and $\tilde{Q}$ such that $\tilde{\mathcal{A}}$ is in general position with respect to $\mathcal{A}$ (see Definition 2.3.4).

We will denote the intersection poset of the arrangement $\check{A}:=\mathcal{A}\cup\tilde{A}$ by $\check{Q}$ and so on. The map

$$\begin{aligned}\sigma\colon (Q\times\tilde{Q})_{[n,2n]} &\to \check{Q}_{[0,n]}\\ (u,v)&\mapsto u\cap v\end{aligned}\tag{2.58}$$

is an isomorphism.

2.3.14 Proposition. *In the above situation let $c\in H_*(\Delta Q_{[k,n]},\Delta Q_{[k,n)})$ and $d\in H_*(\Delta\tilde{Q}_{[l,n]},\Delta\tilde{Q}_{[l,n)})$ with $k+l<n$. Then*

$$h_k(c)\bullet\tilde{h}_l(d)=\sum_{i>0}\check{h}_i(r_i)$$

for classes $r_i\in H_*\left(\Delta\check{Q}_{[i,n]},\Delta\check{Q}_{[i,n)}\right)$.

Proof. By Corollary 2.3.12 we have to show $s_*^{\check{A}}(h_k(c) \bullet \tilde{h}_l(d)) = 0$. It will be in doing so that we employ the ideas laid out in the discussion of the real example.

We set $(X, A) := \mathbb{C}P^k \times \mathbb{C}P^l \times \left(\Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]}), \Delta(Q_{[k,n]} \times \tilde{Q}_{[l,n]})_{[0,2n)}\right)$, $D := \{(x,x) \in PV \times PV\}$, $\mathcal{C}D := (PV \times PV) \setminus D$ and use the map g from Definition 2.3.1. We denote the diagonal map $PV \to PV \times PV$ by Δ and define $\bar{g}\colon (g^{-1}[D], g^{-1}[D] \cap A) \to (PV, \bigcup P\mathcal{A} \cup \bigcup P\tilde{\mathcal{A}})$ by $\Delta \circ \bar{g} = g$. Note that in the real example the projection of $g^{-1}[D]$ to the order complex is the set represented by a dotted line in (2.56). We will first show $h_k(c) \bullet \tilde{h}_l(d) \in \operatorname{im} \bar{g}_*$ and then $s_*^{\check{A}} \circ \bar{g}_* = 0$.

There is a commutative diagram

$$\begin{array}{ccc}
H^*(PV \times PV, \mathcal{C}D) \otimes H_*(X, A) & \xrightarrow{\mathrm{id} \otimes g_*} & H^*(PV \times PV, \mathcal{C}D) \otimes H_*\left((PV, \bigcup P\mathcal{A}) \times (PV, \bigcup P\tilde{\mathcal{A}})\right) \\
\downarrow{\scriptstyle g^* \otimes \mathrm{id}} & & \\
H^*(X, X \setminus g^{-1}[D]) \otimes H_*(X, A) & & \downarrow{\scriptstyle \frown} \\
\downarrow{\scriptstyle \frown} & & \\
H_*(g^{-1}[D], g^{-1}[D] \cap A) & \xrightarrow{g_*} & H_*\left(D, D \cap (\bigcup P\mathcal{A} \times PV \cup PV \times \bigcup P\tilde{\mathcal{A}})\right) \\
& \searrow{\scriptstyle \bar{g}_*} & \cong \uparrow{\scriptstyle \Delta_*} \\
& & H_*(PV, \bigcup P\mathcal{A} \cup \bigcup P\tilde{\mathcal{A}}).
\end{array}$$

Regarding the existence of the cap products in this diagram and commutativity, note that we are entirely dealing with algebraic sets and polynomial maps. Now, if $\vartheta \in H^*(PV \times PV, \mathcal{C}D)$ is the Thom class determined by $\vartheta \frown [PV \times PV] = \Delta_*([PV])$, then

$$\begin{aligned}
h_k(c) \bullet \tilde{h}_l(d) &= \Delta_!(h_k(c) \times \tilde{h}_l(d)) \\
&= \Delta_*^{-1}\left(\vartheta \frown (h_k(c) \times \tilde{h}_l(d))\right) \\
&= \Delta_*^{-1}\left(\vartheta \frown g_*([\mathbb{C}P^k] \times [\mathbb{C}P^l] \times (c \times d))\right) \\
&= \Delta_*^{-1}\left(g_*\left(g^*(\vartheta) \frown ([\mathbb{C}P^k] \times [\mathbb{C}P^l] \times (c \times d))\right)\right) \\
&= \bar{g}_*\left(g^*(\vartheta) \frown ([\mathbb{C}P^k] \times [\mathbb{C}P^l] \times (c \times d))\right).
\end{aligned}$$

By construction of g, $\bar{g}\left(x, y, \sum_{j=0}^r \lambda_j(u_j, v_j)\right) \in u_r \cap v_r$. Firstly this implies that $g^{-1}[D]$ misses $\mathbb{C}P^k \times \mathbb{C}P^l \times \Delta(Q \times \tilde{Q})_{[0,n)}$, and secondly from the reformulation $s^{\check{A}}(\bar{g}(x,y,z)) \leq \sigma\left(s^{Q \times \tilde{Q}}(z)\right)$, where σ is the isomorphism from (2.58), and

Lemma 2.3.7 it can be seen that the diagram

$$\begin{array}{ccc}
(g^{-1}[D], g^{-1}[D]\cap A) & \xrightarrow{s^{\check{\mathcal{A}}}\circ \bar{g}} & (\check{Q}_{[0,n]}, \check{Q}_{[0,n)}) \\
 & \searrow^{\pi} & \uparrow \sigma \\
 & & \big((Q\times\tilde{Q})_{[n,2n]}, (Q\times\tilde{Q})_{[n,2n)}\big) \\
 & & \nearrow_{s^{Q\times\tilde{Q}}} \\
\multicolumn{3}{c}{\big(\Delta(Q_{[k,n]}\times\tilde{Q}_{[l,n]})\setminus\Delta(Q\times\tilde{Q})_{[0,n)}, \Delta(Q_{[k,n]}\times\tilde{Q}_{[l,n]})_{[0,2n)}\setminus\Delta(Q\times\tilde{Q})_{[0,n)}\big),}
\end{array}$$

where π denotes projection onto the third factor, is homotopy commutative. The two arrows on the right hand side of the diagram should be compared to (2.57).

By Proposition 2.3.5 and because the subcomplex $\Delta\big((Q_{[k,n]}\times\tilde{Q}_{[l,n]})\cap(Q\times\tilde{Q})_{[0,n)}\big)$ has dimension at most $n-1-k-l$, we may assume that $\pi[g^{-1}[D]]$ will not only miss this subcomplex, but any cone over it. Therefore π factorizes over the pair

$$\begin{aligned}\big(\Delta(Q_{[k,n]}\times\tilde{Q}_{[l,n]})\setminus\Delta((Q\times\tilde{Q})_{[0,n)}\cup\{(V,V)\}),&\\ \Delta(Q_{[k,n]}\times\tilde{Q}_{[l,n]})_{[0,2n)}\setminus\Delta(Q\times\tilde{Q})_{[0,n)}\big).&\end{aligned}$$

This pair is homeomorphic to $(\Delta(Q_{[k,n]}\times\tilde{Q}_{[l,n]})_{[0,2n)}\setminus\Delta(Q\times\tilde{Q})_{[0,n)})\times([0,1),\{0\})$ and has trivial homology. So $(s^{\check{\mathcal{A}}}\circ\bar{g})_* = s_*^{Q\times\tilde{Q}}\circ\pi_* = s_*\circ 0 = 0$. □

2.3.15 Proposition. *In the above situation let* $c\in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ *and* $d\in H_*(\Delta\tilde{Q}_{[l,n]}, \Delta\tilde{Q}_{[l,n)})$ *with* $k+l<n$*. Then*

$$h_k(c)\bullet\tilde{h}_l(d) = 0.$$

Proof. We choose a hyperplane H in V in general position with respect to the arrangement $\check{A} = \mathcal{A}\cup\tilde{\mathcal{A}}$ and use notation as in Section 2.1. By Proposition 2.1.35 and induction on the dimension of V

$$i_!(h_k(c)\bullet\tilde{h}_l(d)) = i_!(h_k(c))\bullet i_!(\tilde{h}_l(d)) = h^H_{k-1}(\eta_*(c))\bullet\tilde{h}^H_{l-1}(\eta_*(d)) = 0,$$

since the arrangement A^H is again in general position with respect to the arrangement $\tilde{A}^H$ and $(k-1)+(l-1)<n-1$. This implies $h_k(c)\bullet\tilde{h}_l(d)\in\ker i_! = \operatorname{im}\check{h}_0$ by Proposition 2.1.35. But $h_k(c)\bullet\tilde{h}_l(d)\in\bigoplus_{j>0}\operatorname{im}\check{h}_j$ by Proposition 2.3.14 and hence $h_k(c)\bullet\tilde{h}_l(d)=0$. □

Proof of Proposition 2.3.13. We choose a neighbourhood U of $\bigcup P\mathcal{A}$ such that the inclusion $(PV, \bigcup P\mathcal{A})\to(PV,U)$ is a homotopy equivalence. We then choose a copy $\tilde{\mathcal{A}}$ of $\mathcal{A}$ also contained in U, in general position with respect to $\mathcal{A}$ and such that the diagram

$$\begin{array}{ccc}
\mathbb{C}P^i\times(\Delta Q_{[i,n]}, \Delta Q_{[i,n)}) & \xrightarrow{\tilde{f}^i} & (PV, \bigcup P\tilde{\mathcal{A}}) \\
\downarrow f^i & & \downarrow \text{incl.} \\
(PV, \bigcup P\mathcal{A}) & \xrightarrow{\text{incl.}} & (PV, U)
\end{array}$$

commutes up to homotopy. Because of the commutativity of

$$\begin{array}{ccc}
H_*(PV, \bigcup P\mathcal{A}) \otimes H_*(PV, \bigcup P\mathcal{A}) & \xrightarrow{\ \bullet\ } & H_*(PV, \bigcup P\mathcal{A}) \\
\uparrow{\scriptstyle h_k \otimes h_l} & & {\scriptstyle \mathrm{incl}_*}\downarrow{\scriptstyle \cong} \\
H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \otimes H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)}) & & H_*(PV, U) \\
\downarrow{\scriptstyle h_k \otimes \tilde{h}_l} & & {\scriptstyle \mathrm{incl}_*}\uparrow \\
H_*(PV, \bigcup P\mathcal{A}) \otimes H_*(PV, \bigcup P\tilde{\mathcal{A}}) & \xrightarrow{\ \bullet\ } & H_*(PV, \bigcup P\mathcal{A} \cup \bigcup P\tilde{\mathcal{A}})
\end{array}$$

the result follows from Proposition 2.3.15 □

Proof of Theorem 2.2.1. The theorem follows directly from Proposition 2.3.13 and Proposition 2.2.10. □

2.4 Projective c-arrangements

Descriptions of the cohomology ring of the complement of a linear arrangement in terms of generators and relations go back to Arnol'd [Arn69] who gave such a description for the classifying space of the coloured braid group, which is the complement of a complex hyperplane arrangement. He also conjectured a similar formula for general complex hyperplane arrangements, later to be proved by Orlik and Solomon [OS80] (see Remark 2.4.4). Since then several such results on other classes of linear arrangements have been obtained. The approach most useful to us is that of Yuzvinsky, who derived from the complex case of Theorem 2.2.3 (with rational coefficients) a description in terms of generators and relations of the cohomology ring of the complement of a complex linear arrangement with geometric intersection lattice [Yuz99]. Generalizations of his results to real ≥ 2-arrangements and integral coefficients have been stated in [dLS01].

A presentation of the cohomology ring

We will now use a route similar to Yuzvinsky's to obtain from Theorem 2.2.1 a simple description of the cohomology of the complement of a complex projective c-arrangement. These probably form the simplest class of arrangements that still yield a proper generalization of the classical result on complex hyperplane arrangements in this way. The result presented in this section complements results of Feichtner and Ziegler in [FZ00].

2.4.1 Definition. For a positive integer c, we call $\mathcal{A}$ a c-arrangement, if every $A \in \mathcal{A}$ is a subspace of codimension c and $d(q)$ is an integral multiple of c for every $q \in Q$.

2.4.2 Definition. We call a subset M of $\mathcal{A}$ independent, if $n - d\left(\bigcap M\right) = \sum_{A \in M}(n - d(A))$, dependent, if it is not independent, and minimally dependent, if it is dependent but all of its proper subsets are independent.

We will assume $\mathbb{K} = \mathbb{C}$ in this section. Our goal is the following.

2.4.3 Theorem. *Let $\mathcal{A}$ be a complex c-arrangement, $|\mathcal{A}| - 1 =: t \geq 0$, $\mathcal{A} = \{A_0, \ldots, A_t\}$. Let R be the free graded commutative (in the graded sense) ring over the set of generators $\{x\} \cup \{y_i : 1 \leq i \leq t\}$ with $|x| = 2$, $|y_i| = 2c - 1$. Let I be the ideal generated by*

$$\left\{ \sum_{j=0}^{r} (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} : i_0 < \cdots < i_r,\ \left\{A_{i_j}\right\} \text{ is minimally dependent.} \right\}$$
$$\cup \left\{ y_{i_1} \cdots y_{i_r} : i_1 < \cdots < i_r,\ \{A_0\} \cup \left\{A_{i_j}\right\} \text{ is minimally dependent. } \right\}$$
$$\cup \left\{x^c\right\}.$$

The map

$$\begin{aligned} \pi \colon R &\to H^*\left(PV \setminus \bigcup P\mathcal{A}\right), \\ x &\mapsto P(h_{n-1}([\langle V \rangle])), \\ y_i &\mapsto P(h_{n-c}([\langle A_i, V \rangle - \langle A_0, V \rangle])), \end{aligned}$$

where $P \colon H_*(PV, \bigcup P\mathcal{A}) \xrightarrow{\cong} H^*\left(PV \setminus \bigcup P\mathcal{A}\right)$ denotes Poincaré duality, is an epimorphism and $\ker \pi = I$.

We now fix the arrangement $\mathcal{A} = \{A_0, \ldots, A_t\}$.

2.4.4 Remark. For $c = 1$ the complement $PV \setminus \bigcup P\mathcal{A}$ can be regarded as the complement in the affine space $PV \setminus PA_0$ of the linear hyperplane arrangement $\mathcal{A}' := \{PA_i \setminus PA_0 : 1 \leq i \leq t\}$. In this case, the generator x and the corresponding relation can be omitted.

If A_0 is in general position with respect to $\mathcal{A} \setminus \{A_0\}$, the second kind of generators does not occur. This is in particular the case if the arrangement $\mathcal{A}'$ is central, i.e. if $\bigcap \mathcal{A}' \neq \emptyset$. In this case the theorem reduces to the description of the cohomology ring of the complement of $\mathcal{A}'$ given by Orlik and Solomon.

The atomic complex

We now turn to the proof of the theorem. When using simplicial chain complexes, we will always use the complex of non-degenerate simplices and view it as the complex of all simplices modulo degenerate simplices if necessary.

2.4.5 Definition. For an integer k with $0 \leq k \leq n$, we define S_k to be the simplicial complex which has the vertex set $\{0, \ldots, t\}$ and as simplices the sets $I \subset \{0, \ldots, t\}$ with $d\left(\bigcap_{i \in I} A_i\right) \geq k$. This is the *atomic complex* of $Q_{[k,n]}$. We also define D^k to be the reduced ordered (using the natural order of $\{0, \ldots, t\}$) simplicial chain complex of S_k shifted by one, i.e. $D^k_r = \tilde{C}_{r-1}(S_k)$ and in particular $D^k_0 \cong \mathbb{Z}$ generated by the empty simplex.

As is well known, the atomic complex and the order complex, of $Q_{[k,n)}$ in this case, are homotopy equivalent. We describe a homotopy equivalence to fix a concrete isomorphism between their homology groups. Before doing this, we state a useful lemma.

2.4.6 Lemma. *Let P_0, P_1 be posets, $P_i' \subset P_i$. If $f, g\colon (P_0, P_0') \to (P_1, P_1')$ are order preserving functions such that $f(p) \le g(p)$ for all $p \in P_0$, then the maps $f, g\colon (\Delta P_0, \Delta P_0') \to (\Delta P_1, \Delta P_1')$ are homotopic.*

Proof. The map $H\colon \{0,1\} \times P_0 \to P_1$ defined by $H(0,x) := f(x)$, $H(1,x) := g(x)$ is order preserving and hence yields the desired homotopy

$$I \times (\Delta P_0, \Delta P_0') \approx \big(\Delta(\{0,1\} \times P_0), \Delta(\{0,1\} \times P_0')\big) \xrightarrow{H} (\Delta P_1, \Delta P_1'),$$

where we view $\{0,1\}$ as a poset. □

2.4.7 Remark. This lemma is a special case of [Seg68, Prop. 2.1] which is proved in the same way.

2.4.8 Definition and Proposition. *We denote the face poset of S_k by FS_k, but order it by $M \le M'$ if M' is a face of M, that is if $M' \subset M$. We also set $\tilde{F}S_k := FS_k \cup \{\emptyset\}$. The map*

$$\begin{aligned} s\colon (\tilde{F}S_k, FS_k) &\to (Q_{[k,n]}, Q_{[k,n)}) \\ M &\mapsto \bigcap \{A_i\colon i \in M\} \end{aligned}$$

is then order preserving and moreover satisfies $s(M \wedge M') = s(M \cup M') = s(M) \cap s(M') = s(M) \wedge s(M')$, if one side, and therefore the other, exists.

With these definitions, the map $s\colon (\Delta\tilde{F}S_k, \Delta FS_k) \to (\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ is a homotopy equivalence.

2.4.9 Remark. ΔFS_k is the barycentric subdivision of S_k, and $\Delta\tilde{F}S_k$ is a cone over ΔFS_k.

Proof. We define an order preserving map

$$\begin{aligned} r\colon (Q_{[k,n]}, Q_{[k,n)}) &\to (\tilde{F}S_k, FS_k), \\ q &\mapsto \{i\colon A_i \supset q\}. \end{aligned}$$

We have $s(r(q)) \ge q$ for $q \in Q_{[k,n]}$ and $r(s(i)) \le i$ for $i \in \tilde{F}S_k$. Hence, by the preceding lemma r is a homotopy inverse to s, when both maps are regarded as simplicial maps between order complexes. □

2.4.10 Definition and Proposition. *We define chain maps*

$$\begin{aligned} f^k\colon D_r^k &\to C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \\ \langle i_1, \dots, i_r\rangle &\mapsto \langle A_{i_1}, V\rangle \hat{\times} \cdots \hat{\times} \langle A_{i_r}, V\rangle. \end{aligned}$$

For $r = 0$ this is to be understood as $f^k(\langle\rangle) = \langle V\rangle$.

2.4.11 Notation. To simplify the following calculations, we set set

$$\alpha_i := \langle A_i, V\rangle \in C_1(\Delta Q_{[n-c,n]})$$

and sometimes write the multiplication $\hat{\times}$ as juxtaposition.

Proof. To see that f^k is well-defined, we have to check that the right hand side is in $C_*(\Delta Q_{[k,n]})$. But $d\,(A_{i_1} \wedge \cdots \wedge A_{i_r}) \geq k$ by definition of S_k and hence D^k_r.

To see that f^k is a chain map, we calculate

$$\begin{aligned}
\mathfrak{d}(f^k(\langle i_1, \ldots, i_r\rangle)) &= \sum_{j=1}^{r} (-1)^{j+1} \langle A_{i_1}, V\rangle \,\hat{\times} \cdots \hat{\times}\, \mathfrak{d}\langle A_{i_j}, V\rangle \,\hat{\times} \cdots \hat{\times} \langle A_{i_r}, V\rangle \\
&= \sum_{j=1}^{r} (-1)^{j+1} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \langle V\rangle \alpha_{i_{j+1}} \cdots \alpha_{i_r} \\
&\qquad - \sum_{j=1}^{r} (-1)^{j+1} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \langle A_{i_j}\rangle \alpha_{i_{j+1}} \cdots \alpha_{i_r} \\
&= \sum_{j=1}^{r} (-1)^{j+1} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \hat{\alpha}_{i_j} \alpha_{i_{j+1}} \cdots \alpha_{i_r} \\
&\qquad - \sum_{j=1}^{r} (-1)^{j+1} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \langle A_{i_j}\rangle \alpha_{i_{j+1}} \cdots \alpha_{i_r}.
\end{aligned}$$

The first summand equals $f^k(\mathfrak{d}\langle i_1, \ldots, i_r\rangle)$. Since $\langle A_i\rangle \in \mathbb{C}_0(Q_{[n-c,n)})$, the second summand is in $C_*(\Delta Q_{[k,n-c]}) \subset C_*(\Delta Q_{[k,n)})$. ☐

2.4.12 Proposition. *The induced maps* $f^k_*\colon H(D^k) \to H(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ *are isomorphisms.*

Proof. Defining

$$\begin{aligned}
\bar{f}^k\colon D^k_r &\to C_r(\Delta\tilde{F}S_k, \Delta FS_k) \\
\langle i_1, \ldots, i_r\rangle &\mapsto \langle \{i_1\}, \emptyset\rangle \,\hat{\times} \cdots \hat{\times} \langle \{i_r\}, \emptyset\rangle
\end{aligned}$$

the diagram

$$\begin{array}{ccccc}
 & & H_r(D^k) & & \\
 & \swarrow^{f^k_*} & \downarrow^{\bar{f}^k_*} & \searrow^{sd_*}_{\cong} & \\
H_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) & \xleftarrow[\cong]{s_*} & H_r(\Delta\tilde{F}S_k, \Delta FS_k) & \xrightarrow[\cong]{\mathfrak{d}} & \tilde{H}_{r-1}(\Delta FS_k)
\end{array}$$

commutes, where sd is the barycentric subdivision map $\tilde{C}_*(S_k) \to \tilde{C}_*(\Delta FS_k)$. The connecting homomorphism is an isomorphism, because $\tilde{F}S_k$ has the maximum $\emptyset$. The map s_* is an isomorphism because of Proposition 2.4.8. It follows that f^k_* is an isomorphism. ☐

Proof of the presentation

The chain maps f^k would be more useful in a situation in which the chains $\langle A_i, V\rangle$ are cycles. For example, think of affine arrangements, where $(\Delta Q_{[k,n]}, \Delta Q_{[k,n)} \cup \Delta Q_{(k,n]})$ takes the place of $(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$. In our situation they are not. The chains $\langle A_i, V\rangle - \langle A_j, V\rangle$ however are cycles, we will therefore replace the maps f^k by the following maps.

2.4.13 Definition and Proposition. *For a c-arrangement $\mathcal{A}$, we define chain maps*

$$
\begin{aligned}
g^k \colon D_r^k &\to C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \\
\langle i_1, \dots, i_r\rangle &\mapsto \begin{cases} ((\langle A_{i_1}, V\rangle - \langle A_0, V\rangle) \hat{\times} \cdots \hat{\times} (\langle A_{i_r}, V\rangle - \langle A_0, V\rangle), & r = a, \\ 0, & r \neq a, \end{cases}
\end{aligned}
$$

where a is defined by $n-(a+1)c < k \leq n - ac$.

Proof. We check that g^k is a well-defined chain map. For $r > a$ we have $C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \cong 0$, since $n - k < (a+1)c \leq rc$. So we just have to show that $g^k(\langle i_1, \dots, i_a\rangle)$ is a cycle in $C_a(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$. This is true, because each $\langle A_i, V\rangle - \langle A_0, V\rangle$ is a cycle in $C_1(\Delta Q_{[n-c,n]}, \Delta Q_{[n-c,n)})$ and $n - ac \geq k$. □

2.4.14 Proposition. *The maps f^k and g^k are chain homotopic.*

Proof. We define

$$
\begin{aligned}
K \colon D_r^k &\to C_{r+1}(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \\
\langle i_1, \dots, i_r\rangle &\mapsto \begin{cases} f^k(\langle 0, i_1, \dots, i_r\rangle), & r < a, \\ 0, & r \geq a. \end{cases}
\end{aligned}
$$

The right hand side is well defined, because for $r < a$ we have

$$
d(A_0 \cap A_{i_1} \cap \dots \cap A_{i_r}) \geq n - (r+1)c \geq n - ac \geq k.
$$

We calculate $K\mathfrak{d} + \mathfrak{d}K$.

For $r < a$:

$$
\begin{aligned}
&(K\mathfrak{d} + \mathfrak{d}K)\langle i_1, \dots, i_r\rangle \\
&= f^k\left(\sum_{j=1}^{r} (-1)^{j+1} \langle 0, i_1, \dots, \hat{i}_j, \dots, i_r\rangle\right) + \mathfrak{d} f^k(\langle 0, i_1, \dots, i_r\rangle) \\
&= f^k\left(\sum_{j=1}^{r} (-1)^{j+1} \langle 0, i_1, \dots, \hat{i}_j, \dots, i_r\rangle + \mathfrak{d}\langle 0, i_1, \dots, i_r\rangle\right) \\
&= f^k(\langle i_1, \dots, i_r\rangle) = (f^k - g^k)\langle i_1, \dots, i_r\rangle.
\end{aligned}
$$

For $r = a$: We first calculate

$$\begin{aligned} g^k(\langle i_1, \ldots, i_a \rangle) &= (\alpha_{i_i} - \alpha_0) \cdots (\alpha_{i_a} - \alpha_0) \\ &= \alpha_{i_1} \cdots \alpha_{i_a} - \sum_{j=1}^{a} \alpha_{i_1} \cdots \alpha_{i_{j-1}} \alpha_0 \alpha_{i_{j+1}} \cdots \alpha_{i_a} \\ &= \alpha_{i_1} \cdots \alpha_{i_a} + \sum_{j=1}^{a} (-1)^j \alpha_0 \alpha_{i_1} \cdots \hat{\alpha}_{i_j} \cdots \alpha_{i_a} \end{aligned}$$

and with this

$$\begin{aligned} (K\mathfrak{d} + \mathfrak{d}K)\langle i_1, \ldots, i_a \rangle &= f^k \left(\sum_{j=1}^{a} (-1)^{j+1} \langle 0, i_1, \ldots, \hat{i}_j, \ldots, i_a \rangle \right) \\ &= \sum_{j=1}^{a} (-1)^{j+1} \alpha_0 \alpha_{i_1} \cdots \hat{\alpha}_{i_j} \cdots \alpha_{i_a} \\ &= (f^k - g^k)\langle i_1, \ldots, i_a \rangle. \end{aligned}$$

For $r > a$ we have $(K\mathfrak{d} + \mathfrak{d}K)\langle i_1, \ldots, i_r \rangle = 0 = (f^k - g^k)\langle i_1, \ldots, i_r \rangle$, since $C_r(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \cong 0$ as noted before. □

2.4.15 Proposition. *The map π is surjective.*

Proof. By Proposition 2.4.12, Proposition 2.4.14, and of course Proposition 2.1.25, $H^*(PV \setminus \bigcup P\mathcal{A})$ is additively generated by the elements $P\left(h_k\left([g^k\langle i_i, \ldots, i_r \rangle]\right)\right)$ with $k \leq n - rc$. By Theorem 2.2.1

$$\begin{aligned} &P\left(h_k\left([g^k\langle i_i, \ldots, i_r \rangle]\right)\right) \\ &\quad = P(h_k([(\alpha_{i_1} - \alpha_0) \hat{\times} \cdots \hat{\times} (\alpha_{i_r} - \alpha_0)])) \\ &\quad = P(h_{n-1}([\langle V \rangle]))^{n-k-rc} P(h_{n-c}([\alpha_{i_1} - \alpha_0])) \cdots P(h_{n-c}([\alpha_{i_r} - \alpha_0])) \\ &\quad = \pi(x)^{n-k-rc} \pi(y_{i_1}) \cdots \pi(y_{i_r}). \end{aligned}$$

This shows that π is surjective. □

2.4.16 Proposition. $I \subset \ker \pi$.

Proof. First of all

$$\begin{aligned} \pi(x^c) &= P(h_{n-1}([\langle V \rangle]))^c \\ &= P(h_{n-c}([\langle V \rangle])) \\ &= P(h_{n-c}([\mathfrak{d}\langle A_0, V \rangle])) = P(h_{n-c}(0)) = 0. \end{aligned}$$

If $\{A_{i_0},\ldots,A_{i_r}\}$ is minimally dependent, then $d\left(\bigcap_j A_{i_j}\right) = n - rc$ and

$$\begin{aligned}
0 &= P(h_{n-rc}(g_*^{n-rc}([\mathfrak{d}\langle i_0,\ldots,i_r\rangle]))) \\
&= (P\circ h_{n-rc}\circ g_*^{n-rc})\left(\left[\sum_{j=0}^{r}(-1)^j\langle i_0,\ldots,\hat{i}_j,\ldots,i_r\rangle\right]\right) \\
&= \pi\left(\sum_{j=0}^{r}(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r}\right)
\end{aligned}$$

and similarly if $\{A_0, A_{i_1},\ldots,A_{i_r}\}$ is minimally dependent, then

$$\begin{aligned}
0 &= P(h_{n-rc}(g_*^{n-rc}([\mathfrak{d}\langle 0,i_1,\ldots,i_r\rangle]))) \\
&= (P\circ h_{n-rc}\circ g_*^{n-rc})\left(\left[\langle i_1,\ldots,i_r\rangle + \sum_{j=1}^{r}(-1)^j\langle 0,i_1,\ldots,\hat{i}_j,\ldots,i_r\rangle\right]\right) \\
&= \pi\left(y_{i_1}\cdots y_{i_r}\right)
\end{aligned}$$

as claimed. □

2.4.17 Lemma. *If $i_0 < \cdots < i_r$ and $\left\{A_{i_j}\right\}$ is dependent, then $y_{i_0}\cdots y_{i_r}\in I$ and $\sum_j(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r}\in I$.*

Proof. Let $\{A_{i_0},\ldots,A_{i_r}\}$ be dependent. To show $y_{i_0}\cdots y_{i_r}\in I$ we may assume that the set is minimally dependent. Then $\sum_j(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r}\in I$ and $y_{i_0}\left(\sum_j(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r}\right) = y_{i_0}\cdots y_{i_r}$ since $y_0^2 = 0$.

For the second part of the lemma we may assume that $\left\{A_{i_j} : j\le s\right\}$ is minimally dependent. Then

$$\sum_j(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_r} = \underbrace{\left(\sum_{j=0}^{s}(-1)^j y_{i_0}\cdots\hat{y}_{i_j}\cdots y_{i_s}\right)}_{\in I} y_{i_{s+1}}\cdots y_{i_r} + \underbrace{y_{i_0}\cdots y_{i_s}}_{\in I}\sum_{j=s+1}^{r}(-1)^j y_{i_{s+1}}\cdots\hat{y}_{i_j}\cdots y_{i_r}\in I$$

as claimed. □

2.4.18 Proposition. $\ker\pi\subset I$.

Proof. Let $z\in\ker\pi$. We want to show $z\in I$. We may assume that z is a linear combination of elements $x^s y_{i_1}\cdots y_{i_r}$ with $0\le s<c$, $i_1<\cdots<i_r$ and $\{A_{i_1},\ldots,A_{i_r}\}$ independent. Since $\pi(x^s y_{i_1}\cdots y_{i_r})\in\mathrm{im}(P\circ h_{n-cr-s})$ and r and s

are determined by $cr+s$, we may assume z to be homogenous in r and s, i.e. $z = x^s \sum_{i_1<\cdots<i_r} \lambda_i y_{i_1} \cdots y_{i_r}$. We set $k := n - cr - s$. The chain

$$z' := \sum_{i_1<\cdots<i_r} \lambda_i \langle i_1, \dots, i_r \rangle + \sum_{i_1<\cdots<i_r} \lambda_i \sum_{j=1}^{r} (-1)^j \langle 0, i_1, \dots, \hat{i}_j, \dots, i_r \rangle$$

is a cycle in D_r^k (the second summand is a cone over the boundary of the first summand), $0 = \pi(z) = (P \circ h_k)\left(g_*^k([z'])\right)$, and therefore $[z'] = 0$ by Proposition 2.4.12 and Proposition 2.4.14, i.e. z' is a boundary in D^k, which means that there exist μ_i, ν_i such that

$$z' = \mathfrak{d} \left(\sum_{i_1<\cdots<i_r} \mu_i \langle 0, i_1, \dots, i_r \rangle + \sum_{i_0<\cdots<i_r} \nu_i \langle i_0, \dots, i_r \rangle \right)$$

and with $d\left(\bigcap_j A_{i_j} \cap A_0\right) \geq k$ and therefore $\{A_0\} \cup \left\{A_{i_j}\right\}$ dependent for $\mu_i \neq 0$ and $\left\{A_{i_j}\right\}$ dependent for $\nu_i \neq 0$. Comparing coefficients and sorting the simplices by whether the first vertex is 0 yields

$$z = \sum_i \mu_i y_{i_1} \cdots y_{i_r} + \sum_i \nu_i \sum_j (-1)^j y_{i_0} \cdots \hat{y}_{i_j} \cdots y_{i_r} \in I$$

as claimed. ☐

This completes the proof of Theorem 2.4.3 ☐

Bibliography

[Arn69] ARNOL'D, V. I. The cohomology ring of the colored braid group. *Math. Notes*, **5**:138–140, 1969.

[BV73] BOARDMAN, J. M. and VOGT, R. M. *Homotopy Invariant Algebraic Structures on Topological Spaces*, vol. 347 of *Lecture notes in mathematics*. Springer-Verlag, 1973.

[DCP95] DE CONCINI, C. and PROCESI, C. Wonderful models of subspace arrangements. *Selecta Mathematica, New Series*, **1**:459–494, 1995.

[DGM00] DELIGNE, P., GORESKY, M., and MACPHERSON, R. L'algèbre de cohomologie du complément, dans un espace affine, d'une famille finie de sous-espaces affines. *Michigan Math. J.*, **48**:121–136, 2000.

[dLS01] DE LONGUEVILLE, M. and SCHULTZ, C. The cohomology rings of complements of subspace arrangements. *Math. Ann.*, **319**(4):625–646, 2001.

[FF89] FLOYD, E. E. and FLOYD, W. J. Actions of classical small categories, 1989. `http://www.math.vt.edu/people/floyd/research/papers/acsc.html`.

[FZ00] FEICHTNER, E. M. and ZIEGLER, G. M. On cohomology algebras of complex subspace arrangements. *Trans. AMS*, **352**:3523–3555, 2000.

[GM88] GORESKY, M. and MACPHERSON, R. *Stratified Morse Theory*. Springer-Verlag, 1988.

[God58] GODEMENT, R. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris, 1958.

[HV92] HOLLENDER, J. and VOGT, R. M. Modules of topological spaces, applications to homotopy limits and e_∞ structures. *Arch. Math.*, **59**:115–129, 1992.

[OS80] ORLIK, P. and SOLOMON, L. Combinatorics and topology of complements of hyperplanes. *Inventiones math.*, **56**:167–189, 1980.

[Seg68] SEGAL, G. Classifying spaces and spectral sequences. *Publ. Math. IHES*, **34**:105–112, 1968.

[Sha94] Shafarevich, I. R. *Basic Algebraic Geometry*, vol. 1. Springer-Verlag, second edn., 1994.

[Vas01] Vassiliev, V. A. Homology of spaces of knots in any dimensions. *Philosophical Transactions: Mathematical, Physical and Engineering Sciences*, **359**(1784):1343–1364, 2001.

[Vog71] Vogt, R. M. Convenient categories of topological spaces for homotopy theory. *Arch. Math.*, **22**:545–555, 1971.

[WZŽ99] Welker, V., Ziegler, G. M., and Živaljević, R. Homotopy colimits—comparison lemmas for combinatorial applications. *J. reine angew. Math.*, **509**:117–149, 1999.

[Yuz99] Yuzvinsky, S. Rational model of subspace complement on atomic complex. *Publ. Inst. Math., Belgrade*, **66**(80):157–164, 1999.

[Yuz02] —. Small rational model of subspace complement. *Trans. AMS*, **354**:1921–1945, 2002.

[Zie93] Ziegler, G. M. On the difference between real and complex arrangements. *Math. Zeit.*, **212**:1–11, 1993.

[ZŽ93] Ziegler, G. M. and Živaljević, R. Homotopy types of subspace arrangements via diagrams of spaces. *Math. Ann.*, **295**:527–548, 1993.

Zusammenfassung

Diese Dissertationsschrift befasst sich mit Homotopie- und Homologieeigenschaften von Arrangements. Ein *Arrangement* in einem topologischen Raum X ist eine endliche Menge $\mathcal{A}$ von Unterräumen von X. Ein Ziel beim Studium von Arrangements ist die Beschreibung der *Vereinigung* $\bigcup \mathcal{A}$ und des Komplements $X \setminus \bigcup \mathcal{A}$.

Das Hauptergebnis dieser Arbeit ist die Beschreibung des Kohomologierings des Komplements eines Arrangements von linearen Unterräumen eines komplexen projektiven Raumes. Da die additive Struktur des Rings bereits durch Goresky und MacPherson bestimmt wurde, kommt es hier auf die Beschreibung der Produkte an. Dies geschieht durch eine Formel wie sie von Yuzvinsky für komplexe lineare Arrangements angegeben und für rationale Koeffizienten bewiesen wurde. Die Darstellung der Formel ist wie in einer gemeinsamen Arbeit von de Longueville und dem Autor, in der die Formel für den linearen Fall für ganzzahlige Koeffizienten bewiesen und auf gewisse reelle Arrangements verallgemeinert wurde.

Das erste Kapitel behandelt Arrangements in allgemeinen topologischen Räumen. Es werden zuerst kurz Ergebnisse über Diagramme von Räumen dargestellt, die von Ziegler und Živaljević als für das Studium von Homotopieeigenschaften von Arrangements als nützlich erkannt wurden. Wir entwickeln dann eine analoge Theorie von Diagrammen von Kettenkomplexen, um beim Studium von Homologieeigenschaften mehr Freiräume zu haben. Dabei tritt zentral eine Spektralsequenz in Erscheinung, die wir im letzten Abschnitt des ersten Kapitels nutzen, um für Arrangements in Mannigfaltigkeiten eine Produktformel für den Kohomologierings des Komplements herzuleiten. Dadurch, dass sie so allgemein gilt, kann sie aber den Ring nicht vollständig beschreiben. Sie ist gradiert in dem Sinne, dass sie Produkte nur bis auf Terme niedrigerer Filtrierungsstufen im Sinne der Spektralsequenz bestimmt.

Das zweite Kapitel widmet sich linearen Arrangements, affinen wie projektiven. Zunächst werden die Methoden des ersten Kapitels angewandt, um diverse Homologie- und einige Homotopieformeln auf einheitliche Weise zu gewinnen. Dann werden gradierte Produktformeln für affine und projektive Arrangements bewiesen. Es wird gezeigt, dass sich aus ihnen induktiv exakte Formeln gewinnen lassen, wenn sich das Verschwinden von Produkten in gewissen Fällen sicherstellen lässt. Dies ist für affine Arrangements durch ein einfaches geometrisches Argument zu bewerkstelligen. Für projektive Arrangements ist es deutlich schwieriger, dies ist das technische Herz dieser Arbeit. Schließlich wird die Produktformel für projektive Arrangements im Geiste Yuzvinskys genutzt, um aus ihr eine Präsentierung des Kohomologierings nach Orlik-Solomon-Art für eine spezielle Klasse komplexer projektiver Arrangements abzuleiten.