“Combinatorics”
Problem Set 1

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Class homepage:
`http://carsten.codimi.de/comb07/`

1. Introduction

1. For the lottery “6 aus 49”, where six distinct numbers are drawn at random from $\{1, 2, \dots, 49\}$, what is the probability that there are no two “adjacent” numbers $i, i+1$ drawn?

   (Problem by Susanne Olbrisch; see `http://lotto.de/lotto_6aus49_archiv.html` for data.)

2. For an 8×8 chessboard, how many paths are there for the king (who starts at e1$= (5,1)$) that take him to the eighth row in seven steps. (No other pieces are on the board, and the king walks according to usual chess rule, from (i,j) to $(i',j') \in \{i-1, i, i+1\} \times \{j-1, j, j+1\} \cap [8]^2$.

   Extra credit questions:
   How many paths are there that take eight steps?
   How many paths are there for a queen?

3. Show that all the following cardinalities are equal, for $n \geq 1$:
   (i) The number of sequences $(s_1, s_2, \dots, s_{2n}) \in \{-1, +1\}^{2n}$ consisting of n $+1$'s and n -1's such that all the partial sums $\sum_{i=1}^{k} s_i$ are non-negative. (Example: $(1, 1, -1, -1, 1, -1, 1, -1)$.)
   (ii) The number of triangulations of a convex $(n+2)$-gon without additional vertices.
   (iii) Binary parenthesizations of a string of $n+1$ letters. (Example: $((a_1 a_2)((a_3 a_4) a_5))(a_6 a_7)$.)

4. Prove
$$\binom{n}{m}\binom{m}{k} \quad = \quad \binom{n}{k}\binom{n-k}{m-k} \qquad \text{for } n \geq m \geq k \geq 0$$
   by counting pairs of sets (A, B) in two ways, and deduce that
$$\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} \; = \; 2^m \binom{n}{m}.$$
   Extra credit question: Use this to show that
$$\binom{2n}{2k}\binom{2n-2k}{n-k}\binom{2k}{k} \quad = \quad \binom{2n}{n}\binom{n}{k}^2.$$

5. Show that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, $\binom{n}{k} = \frac{k+1}{n-k}\binom{n}{k+1}$ for appropriate n and k, and deduce from this that binomial coefficients are *unimodal*:
$$1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1.$$