"Combinatorics"
Problem Set 10

Prof. Günter M. Ziegler
Dr. Carsten Schultz

Version date: July 3, 2007
Issue date: July 3, 2007
Hand in date: July 11/12, 2007

Class homepage:
`http://carsten.codimi.de/comb07/`

11. Linear Algebra Methods

39. Reverse Odd-town has n inhabitants who are just as fond of clubs as the Odd-Towners. However, in Reverse Odd-Town every club is required to have an even number of members, and any two distinct clubs are required to have an odd number of common members.
    (i) Show that there are at most n clubs if n is odd.
    (ii) Show that there are at most $n-1$ clubs if n is even.
    (iii) Show that these bounds are strict.

40. We examine a random walk on a cyclic graph with n vertices, where in each step one of the two neighbours is chosen with equal probability. Let M be the transition matrix.
    (i) Diagonalize M. What are the eigenspaces of the eigenvalues of absolute value 1?
    Hint: Write $M = \frac{1}{2}(R+L)$, where R is the transition matrix for going to the right at every step and L the transition matrix for going to the left at every step. What is the relation between L and R?
    (ii) Let $p_{i,j}(m)$ denote the probability that in a random walk starting at vertex i the position after m steps is vertex j. Let n be odd. Give an upper bound for $|p_{i,j}(m) - \frac{1}{n}|$ from which it can be seen that $\lim_{m\to\infty} p_{i,j}(m) = \frac{1}{n}$.

41. Let G be a graph with n vertices and Δ_G its Laplacian. Prove that
$$\operatorname{rk} \Delta_G = n - \#\{\text{connected components of } G\}.$$

42. Let G and H be graphs. Their *cartesian product* $G \,\square\, H$ is the graph with vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and (u,v) adjacent to (u',v') iff either $u = u'$ and v is adjacent to v', or $v = v'$ and u is adjacent to u'.
    (i) Let $(x_i)_{i\in V(G)}$ be an eigenvector of Δ_G and $(y_j)_{j\in V(H)}$ an eigenvector of Δ_H. Show that $(x_i y_j)_{(i,j)\in V(G\,\square\, H)}$ is an eigenvector of $\Delta_{G\,\square\, H}$.
    (ii) Show that $\lambda_2(G \,\square\, H) = \min\{\lambda_2(G), \lambda_2(H)\}$.

Should you need a grade or an exam for this class, then please let us know by email.