"Combinatorics"
Problem Set 2

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Class homepage:
`http://carsten.codimi.de/comb07/`

2. Basic Counting

6. We again consider the lottery "6 aus 49". What is the probability that the sum of the drawn numbers is even? What if we would play "5 aus 49" or "6 aus 48"?

7. (i) Prove algebraically that
$$\sum_{k\geq 0} S_{n+1,k+1} x^{\underline{k}} = (x+1)^n.$$
   (ii) Comparing coefficients of appropriate polynomials, derive from this the identity
$$S_{n+1,k+1} = \sum_i \binom{n}{i} S_{i,k}. \tag{1}$$
   (iii) Give a combinatorial argument for the validity of the identity (1).
   (iv) The *Bell numbers* are defined by
$$\mathrm{Bell}(n) = \sum_k S_{n,k}.$$
   What do they count? Derive a recurrence for them from (1) and interpret it combinatorially.

8. Let $i_n^{(r)}$ be the number of permutations of an n-element set with no cycles of length greater than r. State and prove a recurrence for these numbers, where you consider r fixed.

9. For a partition λ, let $f_m(\lambda)$ be the number of times m appears in λ, and let $g_m(\lambda)$ be the number of distinct parts of λ that occur at least m times. For example, $f_2(4333211) = 1$ and $g_2(4333211) = 2$. Show that
$$\sum_{\lambda\in \mathrm{Par}(n)} f_m(\lambda) = \sum_{\lambda\in \mathrm{Par}(n)} g_m(\lambda).$$
   Hint: Both sides of the equation might satisfy the same recurrence.

10. Calculate $\begin{bmatrix}6\\3\end{bmatrix}_q$ as a polynomial in q using the recurrence or q-factorials. List the partitions that each of the coefficients of this polynomial counts according to the theorem discussed in class.