“Combinatorics”
Problem Set 3

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3. Generating Functions

11. Let $G(z) = \sum_{k \geq 0} g_k z^k$ be a formal power series over an arbitrary field. Show that the following statements are equivalent.
    (i) G is invertible, i.e. there is a formal power series H such that $G(z) \cdot H(z) = 1$.
    (ii) $g_0 \neq 0$.

12. Let a_k be the number of k-words over the alphabet $\{\mathrm{n}, \mathrm{w}, \mathrm{e}\}$ with no w next to an e. These words can be interpreted as lattice paths of length k which go north, west, or east and never intersect themselves. Find
    (i) a generating function,
    (ii) a closed form form,
    (iii) a recurrence equation of fixed depth

    for the numbers a_k.

13. The Catalan numbers C_n are defined by

$$C_0 = 1,$$
$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

    Use generating functions to derive a closed form for C_n. Try to make it a nice one.

14. Find an exponential generating function for the number D_n of derangements of an n-set by using the exponential formula or a recurrence equation for D_n. From the exponential generating function derive a formula for D_n that shows that

$$\lim_{n \to \infty} \frac{D_n}{n!} = e^{-1}.$$