"Combinatorics"
Problem Set 6

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6. Posets, Lattices, Möbius functions

23. Let P be a finite poset and $m \in \mathbb{N}$.
    (i) Show that the number of order-preserving maps from P to $\underline{m}$ equals the number of multichains $\hat{0} = I_0 \leq I_1 \leq \cdots \leq I_m = \hat{1}$ of length m in $J(P)$.
    (ii) Is there a similar description of the number of all surjective order-preserving maps from P to $\underline{m}$?
    (iii) Show that the number from (i) also equals the cardinality of $J(P \times \underline{m-1})$.

24. Let P be a locally finite poset.
    (i) Show that $2 - \zeta$ is an invertible element of the incidence algebra of P.
    (ii) Let $x, y \in P$. What does $(2-\zeta)^{-1}(x, y)$ count?

25. Let P and Q be locally finite posets, $p_0, p_1 \in P$, $q_0, q_1 \in Q$. Denote the Möbius functions of P, Q, and $P \times Q$ by μ_P, μ_Q, and $\mu_{P\times Q}$ respectively. Show that

$$\mu_{P\times Q}((p_0, q_0), (p_1, q_1)) = \mu_P(p_0, p_1)\mu_Q(q_0, q_1).$$

26. Let k be a field. For functions $f, g \colon \mathbb{N} \to k$, the following is true: The equality

$$f(n) = \sum_{k\leq n} \binom{n}{k} g(k)$$

holds for all $n \in \mathbb{N}$ if and only if the equality

$$g(n) = \sum_{k\leq n} (-1)^{n-k} \binom{n}{k} f(k)$$

holds for all $n \in \mathbb{N}$.

Derive this proposition from the principle of Möbius inversion by regarding f and g as functions on Boolean lattices which depend only on the ranks of elements.