“Combinatorics”
Problem Set 9

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10. Permutations, Representations

35. Let C_n denote the cyclic group with n elements.
    (i) Determine all irreducible complex representations of C_4.
    (ii) Determine all irreducible complex representations of $C_2 \times C_2$.
    (iii) For one of the above groups, find an irreducible real representation of dimension greater than 1.

36. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n. How many paermutations in $S(n)$ are of cycle type λ?

37. Describe the 2-dimensional irreducible representation of $S(3)$ explicitely by a group homomorphism $S(3) \to U(2)$, where $U(2)$ denotes the group of unitary complex (2×2)-matrices.

38. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n, i.e. $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ and $\sum_i \lambda_i = n$. We define a representation W_λ of $S(n)$ as follows.

    Set $a_i := 1 + \sum_{j<i} \lambda_j$ for $1 \leq i \leq k$. Let
    $$M_\lambda := \{\pi \in S(n) \colon \pi(a_i) < \pi(a_i + 1) < \dots < \pi(a_{i+1} + \lambda_i - 1) \text{ for all } i\}\,.$$
    For example, $M_{(n)} = \{\mathrm{id}\}$, $M_{(1,\dots,1)} = S(n)$ and $|M_{(n-1,1)}| = n$. For $\sigma \in S(n)$ and $\pi \in M_\lambda$, let $\sigma \cdot \pi$ be the unique $\pi' \in M_\lambda$ with
    $$\{\pi'(j) \colon a_i \leq j < a_i + \lambda_i\} = \{(\sigma \circ \pi)(j) \colon a_i \leq j < a_i + \lambda_i\} \quad \text{for all } i.$$
    Denote the induced representation on $\mathbb{C}^{|M_\lambda|}$ by W_λ.
    (i) Which well-known representations are $W_{(n)}$, $W_{(n-1,1)}$, and $W_{(1,\dots,1)}$?
    (ii) For all partitions λ of 4, determine the character χ_{W_λ} of the representation W_λ.
    (iii) Determine all irreducible characters of $S(4)$.
    Hint: Consider the χ_{W_λ} for λ in the order $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$ and use a procedure similar to Gram-Schmidt.