1. PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

Problem 1. Let X be a topological space, $x \in X$ and $n \geq 0$. Show that the following statements are equivalent:

(i) There is an open neighbourhood of x which is homeomorphic to $\mathbb{R}^n$.
(ii) There is a neighbourhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$.

Problem 2. Let $n \geq 0$ and let J be the set $\{0, \ldots, n\} \times \{-1, +1\}$. We define $(U_j, \varphi_j)_{j \in J}$ by

$$U_{k,\sigma} = \{x \in \mathbb{S}^n \colon \sigma x_k > 0\}$$

and

$$\begin{aligned} \varphi_{k,\sigma} \colon U_{k,\sigma} &\to \mathbb{R}^n \\ x = (x_0, \ldots, x_n) &\mapsto (x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n). \end{aligned}$$

Show that $(U_j, \varphi_j)_{j \in J}$ is a C^∞-atlas for $\mathbb{S}^n$.

Problem 3. For $\theta \in \mathbb{R}$ we define

$$\begin{aligned} \varphi_\theta \colon U_\theta := \mathbb{S}^1 \setminus \{(\cos\theta, \sin\theta)\} &\to (\theta, \theta + 2\pi) \subset \mathbb{R} \\ (\cos\rho, \sin\rho) &\mapsto \rho \end{aligned}$$

(i) Explain why this is well-defined and show that $(U_\theta, \varphi_\theta)_{\theta \in \mathbb{R}}$ is a C^∞-atlas for $\mathbb{S}^1$.
(ii) Show that

$$\begin{aligned} \big\{(x_0, x_1) \in \mathbb{S}^1 \colon x_1 > 0\big\} &\to \mathbb{R} \\ x &\mapsto \frac{x_0}{x_1} \end{aligned}$$

is another coordinate chart for $\mathbb{S}^1$ and decide whether it is in the $\mathbb{C}^\infty$-structure defined by the atlas $(U_\theta, \varphi_\theta)_{\theta \in \mathbb{R}}$.

Hand-in date: October 21.