2. PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

Problem 4. Prove the following theorem from class, which defines the product of smooth manifolds.

Let M^m and N^n be smooth manifolds. The topological product $M \times N$ is a topological $(m+n)$-manifold. (You do not have to show this.) Let $(U_i, \varphi_i)_i \in I$ be a smooth atlas for M and $(V_j, \psi_j)_{j \in J}$ a smooth atlas for N. Then $(U_i \times V_j, \varphi_i \times \psi_j)_{(i,j) \in I \times J}$ is a smooth atlas for $M \times N$.

(Here $\varphi_i \times \psi_j \colon U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ with $(\varphi_i \times \psi_j)(x, x') = (\varphi_i(x), \psi_j(x'))$).

Problem 5. Let M_1, M_2, N be smooth manifolds. For $i \in \{1, 2\}$ we define $p_i \colon M_1 \times M_2 \to M_i$ by $p_i(x_1, x_2) = x_i$. Let $f \colon N \to M_1 \times M_2$ be a function. Prove that f is smooth if and only if $p_1 \circ f$ and $p_2 \circ f$ are smooth.

Remark. $M_1 \times M_2$ is the smooth manifold defined in the preceding problem.

Problem 6. Let X, Y be Hausdorff spaces and $f \colon X \to Y$ a continuous map. Prove that the following statements are equivalent.

(i) For every $x \in X$ and every neighbourhood U of x there is a neighbourhood V of $f(x)$ such that $f[X \backslash U] \cap V = \emptyset$.
(ii) f is an embedding.

Remark. If topological spaces make you too nervous, you are allowed to assume that they are metric spaces.

Hand-in date: October 28.