3. PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

Problem 7. On $\mathbb{S}^2 = \big\{(x_0, x_1, x_2)\colon x_0^2 + x_1^2 + x_2^2 = 1\big\}$ we consider coordinates given by the stereographic projection from the north pole

$$y_1 = \frac{x_0}{1 - x_2}, \qquad y_2 = \frac{x_1}{1 - x_2}.$$

Let X, Y be the vector fields defined on $\mathbb{S}^2 \setminus \{0, 0, 1\}$ which are given in these coordinates by

$$X = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}, \qquad Y = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the sterographic projection from the south pole and in the coordinates of Problem 2.

Problem 8. Let M^n be a smooth manifold. For a chart

$$\phi = (\phi^1, \dots, \phi^n)\colon U \to \mathbb{R}^n$$

we define

$$\Phi\colon TU = \bigcup_{p \in U} T_pM \to \mathbb{R}^{2n},$$
$$X_p \mapsto (\phi^1(p), \dots, \phi^n(p), X_p(\phi^1), \dots, X_p(\phi^n)),$$

where $X_p \in T_pM$

(i) Show that Φ is injective and $\Phi[TU] = \phi[U] \times \mathbb{R}^n$.
(ii) Given another chart $\psi\colon V \to \mathbb{R}^n$ and $\Psi\colon TV \to \mathbb{R}^{2n}$ as above, show that
$$\Psi \circ \Phi^{-1}\colon \Phi[T(U \cap V)] \to \Psi[T(U \cap V)]$$
is a diffeomorphism.

Remark. After defining a Hausdorff topology on TM such that for every chart $\phi\colon U \to \mathbb{R}^n$ of M the corresponding map Φ is a homeomorphism of an open subset of TM onto its image (which we will now assume has been done), the functions Φ define a snmooth structure on TM.

From now on (in particular **in the next problem**), we will consider TM as a smooth $2m$-manifold with this structure.

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Problem 9. Let X be a (not necessarily smooth) vector field on the smooth manifold M^n, i.e. a system $(X_p)_{p \in M}$ of tangent vectors $X_p \in T_pM$. Show that the following statements are equivalent.

(i) X is smooth as a function (section) $M \to TM$, $p \mapsto X_p$.

(ii) For every coordinate chart (U, ϕ) and the corresponding coordinate frames $(E_{i,p})$ for T_pM, $p \in U$, the functions $\alpha^i \colon U \to \mathbb{R}$ in

$$X_p = \sum_{i=1}^{n} \alpha^i(p) E_{i,p}$$

are smooth.

(iii) For every smooth $f \colon U \to \mathbb{R}$ defined on an open subset U of M, the function $Xf \colon U \to \mathbb{R}$, $(Xf)(p) = X_p f$ is smooth.