6. PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

Problem 16. Let Δ be a C^∞ n-plane distribution on M^m. Show that the following two statements, which define an involutive distribution, are equivalent.

(i) For every pair of C^∞-vector fields X, Y on M which belong to Δ the vector field $[X, Y]$ belongs to Δ.

(ii) In the neighbourhood of each point of M exist a local basis $X_1, \ldots, X_n$ and C^∞ functions c_{ij}^k such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k \qquad \text{for all } 1 \le i, j \le n.$$

Remark. We have seen later that (i) even implies a stronger version of (ii) with all of the functions c_{ij}^k equal to zero. The point here is that (ii) can be easier to check than (i).

Problem 17. Determine the subset of $\mathbb{R}^2$ on which $\sigma^1 = x^1\,dx^1 + x^2\,dx^2$ and $\sigma^2 = x^2\,dx^1 + x^1\,dx^2$ are linearly independent and find a frame field dual to σ^1, σ^2 over this set.

Problem 18. Show that the restriction of

$$\sigma = x^1\,dx^2 - x^2\,dx^1 + x^3\,dx^4 - x^4\,dx^3$$

from $\mathbb{R}^4$ to the sphere $\mathbb{S}^3$ is nowhere zero.

Problem 19. Formulate and solve an analogue of Problem 8 for T_p^*M instead of T_pM. This should be such that an analogue of the proposition proved in Problem 9 holds, but you need not formulate or prove this.

Hand-in date: November 25.