# 7. PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

**Problem 20.** We consider $\mathbb{S}^2$ with the Riemannian metric inherited from $\mathbb{R}^3$. On $U \subset \mathbb{S}^3$, $U = \left\{(x^1, x^2, x^3) \in \mathbb{S}^2 \colon x^3 > 0\right\}$ we introduce coordinates $(y^1, y^2)$ with $y^1 = x^1, y^2 = x^2$ (compare Problem 2). On $U$ we define vector fields

$$X_1 = \frac{\partial}{\partial y^1}, \qquad X_2 = \frac{\partial}{\partial y^2},$$
$$Y_1 = y^1 \frac{\partial}{\partial y^1} + y^2 \frac{\partial}{\partial y^2}, \qquad Y_2 = -y^2 \frac{\partial}{\partial y^1} + y^1 \frac{\partial}{\partial y^2}.$$

At which points of $U$ are $X_1$ and $X_2$ orthogonal to each other? At which points of $U$ are $Y_1$ and $Y_2$ orthogonal to each other?

**Problem 21.** Let $N^k$ be a closed regular submanifold of $M^n$ and $X$ a smooth vector field on $N$. Prove that $X$ can be extended to a smooth vector field on $M$.

*Hint.* For each $p \in N$ choose a neighborhood $U_x$ of $x$ in $M$ and a preferred coordinate chart $\phi_x \colon U_x \to \mathbb{R}^n$. Use a partition of unity subordinate to the covering of $M$ consisting of the sets $U_x$ and the set $M \setminus N$.

**Definition.** A continuous function $f \colon X \to Y$ is called *proper*, if $f^{-1}[C]$ is compact for every compact $C \subset Y$.

*Remark.* In particular a continuous function $f \colon X \to \mathbb{R}$ is proper if and only if $\{x \in X \colon |f(x)| \le K\}$ is compact for every $K > 0$.

**Problem 22.** Let $M^n$ be a manifold. Prove that there is a proper smooth function $f \colon M \to \mathbb{R}_{\ge 0}$.

*Hint.* Exhaust $M$ by compact sets $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots$ as in the proof of the existence of regular coverings. From this obtain an open covering of $M$ (consider the $T_i$ in that proof) and a partition of unity subordinate to this covering and produce a function which takes only values greater than $i$ on $M \setminus K_i$.

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