

**7. PROBLEM SET FOR “DIFFERENTIAL GEOMETRY II”
AKA “ANALYSIS AND GEOMETRY ON MANIFOLDS”
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Problem 20. We consider \mathbb{S}^2 with the Riemannian metric inherited from \mathbb{R}^3 . On $U \subset \mathbb{S}^3$, $U = \{(x^1, x^2, x^3) \in \mathbb{S}^2: x^3 > 0\}$ we introduce coordinates (y^1, y^2) with $y^1 = x^1, y^2 = x^2$ (compare Problem 2). On U we define vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y^1}, & X_2 &= \frac{\partial}{\partial y^2}, \\ Y_1 &= y^1 \frac{\partial}{\partial y^1} + y^2 \frac{\partial}{\partial y^2}, & Y_2 &= -y^2 \frac{\partial}{\partial y^1} + y^1 \frac{\partial}{\partial y^2}. \end{aligned}$$

At which points of U are X_1 and X_2 orthogonal to each other? At which points of U are Y_1 and Y_2 orthogonal to each other?

Problem 21. Let N^k be a closed regular submanifold of M^n and X a smooth vector field on N . Prove that X can be extended to a smooth vector field on M .

Hint. For each $p \in N$ choose a neighborhood U_x of x in M and a preferred coordinate chart $\phi_x: U_x \rightarrow \mathbb{R}^n$. Use a partition of unity subordinate to the covering of M consisting of the sets U_x and the set $M \setminus N$.

Definition. A continuous function $f: X \rightarrow Y$ is called *proper*, if $f^{-1}[C]$ is compact for every compact $C \subset Y$.

Remark. In particular a continuous function $f: X \rightarrow \mathbb{R}$ is proper if and only if $\{x \in X: |f(x)| \leq K\}$ is compact for every $K > 0$.

Problem 22. Let M^n be a manifold. Prove that there is a proper smooth function $f: M \rightarrow \mathbb{R}_{\geq 0}$.

Hint. Exhaust M by compact sets $\emptyset = K_0 \subset K_1 \subset K_2 \subset \dots$ as in the proof of the existence of regular coverings. From this obtain an open covering of M (consider the T_i in that proof) and a partition of unity subordinate to this covering and produce a function which takes only values greater than i on $M \setminus K_i$.