8. PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

Problem 23. Let $i\colon \mathbb{S}^3 \to \mathbb{R}^4$ denote the inclusion map.

(i) Show that for every $p \in \mathbb{S}^3$ the kernel of the map $i^*\colon T_p^*\mathbb{R}^4 \to T_p^*\mathbb{S}^3$ equals $\big\{\lambda(x^1dx^1 + x^2dx^2 + x^3dx^3 + x^4dx^4)\colon \lambda \in \mathbb{R}\big\}$.

(ii) Use this for another proof that the restriction $i^*\sigma$ of

$$\sigma = x^1\,dx^2 - x^2\,dx^1 + x^3\,dx^4 - x^4\,dx^3$$

to $\mathbb{S}^3$ is nowhere zero.

Problem 24. Let V be a finite-dimensional $\mathbb{R}$-vector space and

$$\phi_1, \ldots, \phi_r \in V^* = \textstyle\bigwedge^1(V).$$

Show that the system $(\phi_1, \ldots, \phi_r)$ is linearly dependent if and only if

$$\phi_1 \wedge \cdots \wedge \phi_r = 0.$$

Problem 25. Let $\theta \in \bigwedge^n(\mathbb{R}^{n+1})$ be defined by

$$\theta = \sum_{k=1}^{n+1} (-1)^k x_k\,dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1}.$$

Let $i\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ denote the inclusion map. Express $i^*\theta \in \bigwedge^n(\mathbb{S}^n)$, the restriction of θ to $\mathbb{S}^n$, in the coordinates of Problem 2.

Remark. The notation $dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1}$ means that dx^k should be omitted, i.e. it is short for $dx^1 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \cdots \wedge dx^{n+1}$.

Expressing an alternating n-form in coordinates given by $\phi\colon U \to \mathbb{R}^n$ in particular means to express it in the basis $d\phi^1 \wedge \cdots \wedge d\phi^n$.

Hand-in date: December 9.