9. PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

Problem 26. Let M be a manifold and $\omega \in \bigwedge^1(M)$. Let $a, b \in \mathbb{R}$, $a < b$, and $F\colon [a,b] \to M$ be a smooth curve. Let $c, d \in \mathbb{R}$, $c < d$, and $g\colon [c,d] \to [a,b]$ be a smooth map with $g(c) = a$ and $g(d) = b$. Prove that

$$\int_{[a,b]} F^*(\omega) = \int_{[c,d]} (F \circ g)^*(\omega).$$

What would we obtain if $g(c) = b$, $g(d) = a$?

Problem 27. Let $M = \mathbb{R}^n \setminus \{0\}$, $\omega \in \bigwedge^{n-1}(M)$, and $F\colon M \to M$, $F(x) = \frac{x}{\|x\|}$. Show that $F^*(\omega)$ is closed.

Problem 28. For $k \in \mathbb{Z}$ let

$$\begin{aligned} f_k\colon \mathbb{S}^1 &\to \mathbb{R}^2 \setminus \{0\} \\ (\cos\phi, \sin\phi) &\mapsto (\cos k\phi, \sin k\phi). \end{aligned}$$

Note that this is well-defined. Also let $\omega \in \bigwedge^1(\mathbb{R}^2 \setminus \{0\})$ be defined by

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

(i) Calculate $\int_{\mathbb{S}^1} f_k^*(\omega)$.

(ii) Let $g\colon \mathbb{D}^2 \to \mathbb{R}^2 \setminus \{0\}$ be a smooth map. Show that $\int_{\mathbb{S}^1} g^*(\omega) = 0$. (Here $\mathbb{D}^2$ is the 2-disk and $\mathbb{S}^1 = \partial\mathbb{D}^2$.)

(iii) For which k is there a smooth map $F\colon \mathbb{D}^2 \to \mathbb{R}^2 \setminus \{0\}$ such that $F|_{\mathbb{S}^1} = f_k$?

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