10. PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

Problem 29. For the covariant derivative $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined for a submanifold M of $\mathbb{R}^n$ supply the proofs of the following two properties, which were omitted in class.

(3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.

(4) $\nabla_X Y - \nabla_Y X = [X, Y] = L_X Y$.

Problem 30. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function and $f(x) > 0$ for all x. Let $M \subset \mathbb{R}^3$ be the submanifold $M := \{(x, y, z)\colon y^2 + z^2 = f(x)^2\}$. Let $x_0 \in \mathbb{R}$. Determine all geodesic curves $p = (p^1, p^2, p^3)$ on M for which $p^1(t) = x_0$ for all t. (The answer will probably depend on the behaviour of f near x_0.)

Problem 31. Let M be a Riemannian manifold and N a submanifold of codimension 1. Let Z be a vector field on M such that Z_q is a unit normal vector to N (i.e. orthogonal to $T_q M$ and with norm 1) for every $q \in N$. Let $p \in N$ and $X_p \in T_p N \subset T_p M$. Show that $\nabla_{X_p} Z \in T_p N$. (Here ∇ is the Riemannian connection on M.)

Hand-in date: January 20.