Problem 29. For the covariant derivative $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined for a submanifold $M$ of $\mathbb{R}^n$ supply the proofs of the following two properties, which were omitted in class.

3. $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$.
4. $\nabla_X Y - \nabla_Y X = [X, Y] = L_X Y$.

Problem 30. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function and $f(x) > 0$ for all $x$. Let $M \subset \mathbb{R}^3$ be the submanifold $M := \{(x, y, z) : y^2 + z^2 = f(x)^2\}$. Let $x_0 \in \mathbb{R}$. Determine all geodesic curves $p = (p^1, p^2, p^3)$ on $M$ for which $p^1(t) = x_0$ for all $t$. (The answer will probably depend on the behaviour of $f$ near $x_0$.)

Problem 31. Let $M$ be a Riemannian manifold and $N$ a submanifold of codimension 1. Let $Z$ be a vector field on $M$ such that $Z_q$ is a unit normal vector to $N$ (i.e. orthogonal to $T_q M$ and with norm 1) for every $q \in N$. Let $p \in N$ and $X_p \in T_p N \subset T_p M$. Show that $\nabla_{X_p} Z \in T_p N$. (Here $\nabla$ is the Riemannian connection on $M$.)

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