EXTRA PROBLEM SET FOR “DIFFERENTIAL GEOMETRY II” AKA “ANALYSIS AND GEOMETRY ON MANIFOLDS”
WINTER TERM 2009/10

Please see the remarks on our website.

**Extra problem 1.** Let \( n \geq 0 \) and \( a : S^n \to S^n \) be the antipodal map \( a(x) = -x \). Let \( \theta \in \bigwedge^n(S^n) \) be the form

\[
\theta = \sum_{k=1}^{n+1} (-1)^k x_k \, dx^1 \wedge \cdots \wedge dx^k \wedge \cdots \wedge dx^{n+1}.
\]

Compare the forms \( \theta \) and \( a^*(\theta) \).

**Extra problem 2.** Give an example of an \( \alpha \in \bigwedge^1(\mathbb{R}^3) \) such that \( d\alpha \) is nowhere zero but \( \alpha \wedge d\alpha = 0 \). For this \( \alpha \) show that the distribution \( \Delta \) defined by

\[
\Delta_p = \{ X \in T_p\mathbb{R}^3 : \alpha(p)(X) = 0 \}
\]

is involutive.

**Extra problem 3.** Let \( M^n \) be a manifold and \( F : M \to \mathbb{R}^m \) an embedding. For a vector \( 0 \neq w \in \mathbb{R}^m \) consider the hyperplane \( H_w = \{ u \in \mathbb{R}^m : \langle u, w \rangle = 0 \} \) (a submanifold of \( \mathbb{R}^m \)) and the orthogonal projection

\[
p_w : \mathbb{R}^m \to H_w,
\]

\[
u \mapsto u - w \frac{\langle u, w \rangle}{\langle w, w \rangle}.
\]

Show that \( p_w \circ F \) is an embedding if and only if the following two conditions are satisfied.

(i) For all \( p, q \in M, p \neq q \), we have \( F(p) - F(q) \notin \mathbb{R}w \).

(ii) For every \( p \in M \) and \( 0 \neq X_p \in T_p(M) \) we have

\[
w \neq F_*(X_p) \in T_{F(p)}\mathbb{R}^m \cong \mathbb{R}^m.
\]

(Slight abuse of notation here.)

**Extra problem 4.** Express the form \( \theta \in \bigwedge^2(S^2) \) of Extra problem 1 in stereographic coordinates (compare Problem 7).

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*Hand-in date: January 9.*
Extra problem 5. Let $A, B \subset M$ be closed subsets of a manifold and $A \cap B = \emptyset$. Show that there is a smooth function $f: M \to \mathbb{R}_{\geq 0}$ such that $f(a) = 1$ for all $a \in A$ and $f(b) = 0$ for all $b \in B$.

*Hint.* Partition of unity.

Extra problem 6. Give an example of a manifold $M$, a point $p \in M$ and vector field $X, Y, Z$ on $M$ such that $X_p = Y_p$ but $(L_X Z)_p \neq (L_Y Z)_p$. 