EXTRA PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS"
WINTER TERM 2009/10

Please see the remarks on our website.

Extra problem 1. Let $n \geq 0$ and $a\colon \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map $a(x) = -x$. Let $\theta \in \bigwedge^n(\mathbb{S}^n)$ be the form

$$\theta = \sum_{k=1}^{n+1} (-1)^k x_k \, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1}.$$

Compare the forms θ and $a^*(\theta)$.

Extra problem 2. Give an example of an $\alpha \in \bigwedge^1(\mathbb{R}^3)$ such that $d\alpha$ is nowhere zero but $\alpha \wedge d\alpha = 0$. For this α show that the distribution Δ defined by

$$\Delta_p = \left\{ X_p \in T_p\mathbb{R}^3 \colon \alpha_p(X_p) = 0 \right\}$$

is involutive.

Extra problem 3. Let M^n be a manifold and $F\colon M \to \mathbb{R}^m$ an embedding. For a vector $0 \neq w \in \mathbb{R}^m$ consider the hyperplane $H_w = \{u \in \mathbb{R}^m \colon \langle u, w \rangle = 0\}$ (a submanifold of $\mathbb{R}^m$) and the orthogonal projection

$$p_w \colon \mathbb{R}^m \to H_w,$$
$$u \mapsto u - w\frac{\langle u, w \rangle}{\langle w, w \rangle}.$$

Show that $p_w \circ F$ is an embedding if and only if the following two conditions are satisfied.

(i) For all $p, q \in M$, $p \neq q$, we have $F(p) - F(q) \notin \mathbb{R}w$.

(ii) For every $p \in M$ and $0 \neq X_p \in T_p(M)$ we have

$$w \neq F_*(X_p) \in T_{F(p)}\mathbb{R}^m \cong \mathbb{R}^m.$$

(Slight abuse of notation here.)

Extra problem 4. Express the form $\theta \in \bigwedge^2(\mathbb{S}^2)$ of Extra problem 1 in stereographic coordinates (compare Problem 7).

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Extra problem 5. Let $A, B \subset M$ be closed subsets of a manifold and $A \cap B = \emptyset$. Show that there is a smooth function $f\colon M \to \mathbb{R}_{\geq 0}$ such that $f(a) = 1$ for all $a \in A$ and $f(b) = 0$ for all $b \in B$.

Hint. Partition of unity.

Extra problem 6. Give an example of a manifold M, a point $p \in M$ and vector field X, Y, Z on M such that $X_p = Y_p$ but $(L_X Z)_p \neq (L_Y Z)_p$.