FROM THE 8$^{\text{TH}}$ PROBLEM SET FOR "DIFFERENTIAL GEOMETRY II" AKA "ANALYSIS AND GEOMETRY ON MANIFOLDS" WINTER TERM 2009/10

Problem 25. Let $\theta \in \bigwedge^n(\mathbb{R}^{n+1})$ be defined by

$$\theta = \sum_{k=1}^{n+1} (-1)^k x_k \, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1}.$$

Let $i\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ denote the inclusion map. Express $i^*\theta \in \bigwedge^n(\mathbb{S}^n)$, the restriction of θ to $\mathbb{S}^n$, in the coordinates of Problem 2.

Solution. We will be more careful with the notation than we usually would be. In particular we will distinguish functions on $\mathbb{R}^{n+1}$ from their restrictions to $\mathbb{S}^n$.

We fix $(\ell, \sigma) \in \{1, \ldots, n+1\} \times \{-1, +1\}$ and consider.

$$\varphi_{\ell,\sigma}\colon U_{\ell,\sigma} = \{x \in \mathbb{S}^n \colon \sigma x_\ell > 0\} \to \mathbb{R}^n$$

$$\varphi^j_{\ell,\sigma} = \begin{cases} x^j \circ i, & j < \ell, \\ x^{j+1} \circ i, & j \geq \ell. \end{cases}$$

We have

$$\left(\sum_j (x^j)^2 \right) \circ i = 1, \tag{1}$$

and applying d yields

$$2 \sum_j (x^j \circ i) \cdot d(x^j \circ i) = 0. \tag{2}$$

Now we have

$$i^* \left(x_\ell \, dx^1 \wedge \cdots \wedge \widehat{dx^\ell} \wedge \cdots \wedge dx^{n+1} \right) = (x^\ell \circ i) d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma}.$$

For $k < \ell$ we obtain using (2)

$$\begin{aligned} & i^* \left(x_\ell \, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1} \right) \\ &\quad = -d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge \widehat{d\varphi^k_{\ell,\sigma}} \wedge \cdots \wedge d\varphi^{\ell-1}_{\ell,\sigma} \wedge \left(\sum_j \varphi^j_{\ell,\sigma} d\varphi^j_{\ell,\sigma} \right) \wedge d\varphi^\ell_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma} \\ &\quad = -d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge \widehat{d\varphi^k_{\ell,\sigma}} \wedge \cdots \wedge d\varphi^{\ell-1}_{\ell,\sigma} \wedge \varphi^k_{\ell,\sigma} d\varphi^k_{\ell,\sigma} \wedge d\varphi^\ell_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma} \\ &\quad = (-1)^{\ell-k} \varphi^k_{\ell,\sigma} d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma}. \end{aligned}$$

Date: December 10.

For $k > \ell$ we obtain

$$i^*\left(x_\ell\, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1}\right)$$
$$= -d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^{\ell-1}_{\ell,\sigma} \wedge \left(\sum_j \varphi^j_{\ell,\sigma} d\varphi^j_{\ell,\sigma}\right) \wedge d\varphi^\ell_{\ell,\sigma} \wedge \cdots \wedge \widehat{d\varphi^{k-1}_{\ell,\sigma}} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma}$$
$$= -d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^{\ell-1}_{\ell,\sigma} \wedge \varphi^{k-1}_{\ell,\sigma} d\varphi^{k-1}_{\ell,\sigma} \wedge d\varphi^\ell_{\ell,\sigma} \wedge \cdots \wedge \widehat{d\varphi^{k-1}_{\ell,\sigma}} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma}$$
$$= (-1)^{k-\ell} \varphi^{k-1}_{\ell,\sigma} d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma}$$

Summarizing, we have

$$i^*\left((-1)^k x_\ell\, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^{n+1}\right) = (-1)^\ell (x^k \circ i) d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma}$$

for all k and hence

$$\begin{aligned} i^*\theta &= (-1)^\ell \sum_{k=1}^{n+1} \frac{(x^k \circ i)^2}{x^\ell \circ i} d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma} \\ &= \frac{(-1)^\ell}{x^\ell \circ i} d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma} \\ &= \sigma \cdot (-1)^\ell \left(1 - \sum_j \left(\varphi^j_{\ell,\sigma}\right)^2\right)^{-1/2} d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma}, \end{aligned}$$

where we have used (1) for the last two equalities. □

Remark. If we want to check whether we got the signs right, it is easy to do so. Let $p = (\varphi_{\ell,\sigma})^{-1}(0)$. We have $x^j(p) = \sigma\delta_{\ell,j}$. Calculation or staring hard reveals that the frame corresponding to $\varphi_{\ell,\sigma}$ satifsfies

$$i_*(E_{j,p}) = \begin{cases} \frac{\partial}{\partial x^j}, & j < \ell, \\ \frac{\partial}{\partial x^{j+1}}, & j \geq \ell. \end{cases}$$

Therefore

$$(i^*\theta)(E_{1,p}, \ldots, E_{n,p}) = \theta_p\left(\frac{\partial}{\partial x^1}, \ldots, \widehat{\frac{\partial}{\partial x^\ell}}, \ldots, \frac{\partial}{\partial x^{n+1}}\right) = (-1)^\ell x_\ell(p) = (-1)^\ell \sigma.$$

This is in accordance with

$$\left(\sigma \cdot (-1)^\ell \left(1 - \sum_j \left(\varphi^j_{\ell,\sigma}\right)^2\right)^{-1/2} d\varphi^1_{\ell,\sigma} \wedge \cdots \wedge d\varphi^n_{\ell,\sigma}\right)(E_{1,p}, \ldots, E_{n,p}) =$$
$$= \sigma \cdot (-1)^\ell \left(1 - \sum_j \left(\varphi^j_{\ell,\sigma}(p)\right)^2\right)^{-1/2} = \sigma \cdot (-1)^\ell.$$