The Lovász Conjecture and extensions
Graph colourings, spaces of edges and spaces of circuits

Carsten Schultz
Technische Universität Berlin

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Outline

1. Main result

2. Examples and Consequences
   - Cycles in complete and cyclic graphs
   - Graph colouring obstructions

3. Bits of the proof

4. Preview of further results
Outline

1 Main result

2 Examples and Consequences
   - Cycles in complete and cyclic graphs
   - Graph colouring obstructions

3 Bits of the proof

4 Preview of further results
Hom-posets and -complexes

- A graph homomorphism $G \to H$ is a function $V(G) \to V(H)$ that preserves the adjacency relation.
- A multi-homomorphism $\phi: G \to H$ is
  \[
  \phi: V(G) \to \mathcal{P}(V(H)) \setminus \{\emptyset\}
  \]
  such that every choice function for $\phi$ is a homomorphism
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  \[ \phi : V(G) \to \mathcal{P}(V(H)) \setminus \{\emptyset\} \]
  such that every choice function for $\phi$ is a homomorphism.
- $\text{Hom}(G, H)$ is the poset of all multi-homomorphisms from $G$ to $H$.
- The composition map
  \[ \star : \text{Hom}(G, G') \times \text{Hom}(G', G'') \to \text{Hom}(G, G'') \]
  \[ (\phi \star \rho)(v) := \rho[\phi(v)] \]
  is associative and order-preserving.
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  is associative and order-preserving.
- $\alpha: G \rightarrow G$ with $\alpha^2 = \text{id}$ makes $|\text{Hom}(G, H)|$ into a free $\mathbb{Z}_2$-space, if $\alpha$ flips an edge and $H$ is loop-free.
- In particular, $|\text{Hom}(K_2, H)|$ is a free $\mathbb{Z}_2$-space.
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  is associative and order-preserving.
- \( \alpha : G \rightarrow G \) with \( \alpha^2 = \text{id} \) makes \( |\text{Hom}(G, H)| \) into a free \( \mathbb{Z}_2 \)-space, if \( \alpha \) flips an edge and \( H \) is loop-free.
- In particular, \( |\text{Hom}(K_2, H)| \) is a free \( \mathbb{Z}_2 \)-space.
- \( |\text{Hom}(K_2, K_n)| \approx_{\mathbb{Z}_2} S^{n-2} \).
Lovász’ Theorem and Conjecture

**Theorem (Lovász ’78)**

Let $G$ be a graph. Then

$$\chi(G) \geq \text{ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| + 2.$$ 

**Theorem (Babson & Kozlov ’04)**

Let $G$ be a graph, $r \geq 1$. Then

$$\chi(G) \geq \text{coind}_{\mathbb{Z}_2} |\text{Hom}(C_{2r+1}, G)| + 3.$$
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**Question**

*What is the relationship between $\text{Hom}(K_2, G)$ and $\text{Hom}(C_{2r+1}, G)$?*
Spaces of cycles of arbitrary lengths

The main theorem

- There is a homomorphism $C_{m+2} \to \mathbb{Z}_2 C_m$.
- This induces $\text{Hom}(C_m, G) \to \mathbb{Z}_2 \text{Hom}(C_{m+2}, G)$.
- We can consider $\text{colim}_r |\text{Hom}(C_{2r+1}, G)|$. 
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- We can consider $\text{colim}_r |\text{Hom}(C_{2r+1}, G)|$.

Theorem

$$\text{colim}_r |\text{Hom}(C_{2r+1}, G)| \cong \mathbb{Z}_2 \text{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, |\text{Hom}(K_2, G)|)$$

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Spaces of cycles of arbitrary lengths in complete graphs

Since $|\text{Hom}(K_2, K_n)| \approx \mathbb{Z}_2 \mathbb{S}^{n-2}$:
- $\text{colim}_r |\text{Hom}(C_{2r}, K_n)| \simeq \text{Map}(\mathbb{S}^1, \mathbb{S}^{n-2})$ free loop space of a sphere.
- $\text{colim}_r |\text{Hom}(C_{2r+1}, K_n)| \simeq \text{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^{n-2})$.

Spaces of cycles of arbitrary lengths
in complete graphs

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There is a canonical embedding map

$$V_{2,n-1} := \{ (x, y) \in \mathbb{S}^{n-2} : \langle x, y \rangle = 0 \} \longrightarrow \text{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^{n-2})$$

“Start at $x$, follow great circle through $y$.”

Spaces of cycles of arbitrary lengths in complete graphs

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“Start at $x$, follow great circle through $y$.”

**Theorem (S, conjectured by Csorba)**

$$|\text{Hom}(C_5, K_n)| \approx V_{2,n-1}.$$

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**Kozlov, D. N.** Cohomology of colorings of cycles, 2005. math.AT/0507117.


**Csorba, P. and Lutz, F. H.** Graph coloring manifolds, 2005. math.CO/0510177.
Spaces of cycles of arbitrary lengths in cyclic graphs

**Proposition**

\[
\left| \text{Hom}(K_2, C_{2r+1}) \right| \approx \mathbb{Z}_2
\]

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\left| \text{Hom}(K_2, C_{2r}) \right| \approx \mathbb{Z}_2
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Spaces of cycles of arbitrary lengths
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Proposition

\[ |\text{Hom}(K_2, C_{2r+1})| \approx \mathbb{Z}_2 \]
\[ |\text{Hom}(K_2, C_{2r})| \approx \mathbb{Z}_2 \]

Corollary

\[ \text{colim}_r |\text{Hom}(C_{2r}, C_m)| \approx \text{Map}(S^1, |\text{Hom}(K_2, C_m)|) \approx \bigsqcup_{\mathbb{Z}} S^1 \]
\[ \text{colim}_r |\text{Hom}(C_{2r+1}, C_m)| \approx \text{Map}_{\mathbb{Z}_2}(S^1, |\text{Hom}(K_2, C_m)|) \approx \begin{cases} \bigsqcup_{\mathbb{Z}} S^1, & \text{if } m \text{ odd,} \\ \emptyset, & \text{if } m \text{ even.} \end{cases} \]
Free $\mathbb{Z}_2$-spaces

Definition

Let $X$ be a free $\mathbb{Z}_2$-space. We define

\[
\text{ind}_{\mathbb{Z}_2} X := \min \left\{ k : \text{There is a } \mathbb{Z}_2\text{-map } X \to S^k \right\},
\]

\[
\text{coind}_{\mathbb{Z}_2} X := \max \left\{ k : \text{There is a } \mathbb{Z}_2\text{-map } S^k \to X \right\},
\]

\[
\text{cohom-ind}_{\mathbb{Z}_2} X := \max \left\{ k : f^*(\gamma^k) \neq 0 \right\},
\]

where $f : X/\mathbb{Z}_2 \to \mathbb{R}P^\infty$ is classifying and $H^*\left(\mathbb{R}P^\infty; \mathbb{Z}_2\right) = \mathbb{Z}_2[\gamma]$. 
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where $f : X/\mathbb{Z}_2 \to \mathbb{RP}^\infty$ is classifying and $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[\gamma]$.

**Properties**

- $\text{conn } X + 1 \leq \text{coind}_\mathbb{Z}_2 X \leq \text{cohom-ind}_\mathbb{Z}_2 X \leq \text{ind}_\mathbb{Z}_2 X$
- $\text{coind}_\mathbb{Z}_2 S^n = \text{ind}_\mathbb{Z}_2 S^n = n$
- If there is $X \to \mathbb{Z}_2 Y$ then $x\text{-ind}_\mathbb{Z}_2 X \leq x\text{-ind}_\mathbb{Z}_2 Y$. 
Some topological facts

Let $X$, $Y$ be free $\mathbb{Z}_2$-spaces and $S^1_b$ the 1-sphere with the $\mathbb{Z}_2$-operations

$$\tau \cdot (x_0, x_1) := (-x_0, -x_1), \quad (x_0, x_k) \cdot \tau := (-x_0, x_1).$$

There is an adjunction

$$\text{Top}^{\mathbb{Z}_2}(Y, \text{Map}_{\mathbb{Z}_2}(S^1_b, X)) \cong \text{Top}^{\mathbb{Z}_2}(S^1_b \times \mathbb{Z}_2 Y, X).$$
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- We obtain inequalities

$$\text{cohom-ind}_{\mathbb{Z}_2} X + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2}(S^1_b \times \mathbb{Z}_2 X),$$

$$\text{ind}_{\mathbb{Z}_2}(S^1_b \times \mathbb{Z}_2 X) \leq \text{ind}_{\mathbb{Z}_2} X + 1,$$

$$\text{cohom-ind}_{\mathbb{Z}_2} \text{Map}_{\mathbb{Z}_2}(S^1_b, X) + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2} X,$$

$$\text{coind}_{\mathbb{Z}_2} X \leq \text{coind}_{\mathbb{Z}_2} \text{Map}_{\mathbb{Z}_2}(S^1_b, X) + 1.$$
Spaces of cycles of arbitrary lengths
Consequences

**Theorems (Lovász '78, Babson & Kozlov '04)**

\[ \chi(G) \geq \text{ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| + 2. \]
\[ \chi(G) \geq \text{coind}_{\mathbb{Z}_2} |\text{Hom}(C_{2r+1}, G)| + 3. \]

**Theorem**

\[ \text{colim}_r |\text{Hom}(C_{2r+1}, G)| \cong_{\mathbb{Z}_2} \text{Map}_{\mathbb{Z}_2} (S^1_b, |\text{Hom}(K_2, G)|). \]
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Consequences

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**Corollary**

\[
\text{cohom-ind}_{\mathbb{Z}_2}|\text{Hom}(C_{2r+1}, G)| + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2}|\text{Hom}(K_2, G)|.
\]
\[
\lim_{r \to \infty} \text{coind}_{\mathbb{Z}_2}|\text{Hom}(C_{2r+1}, G)| + 1 \geq \text{coind}_{\mathbb{Z}_2}|\text{Hom}(K_2, G)|.
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Hom($K_2, C_{2r+1}$)
A closer look.

Reminder

*We want to compare* Hom($K_2, G$) and Hom($C_{2r+1}, G$).
$\text{Hom}(K_2, C_{2r+1})$

A closer look.

**Definition**

$S^1_b$ is the 1-sphere with the $\mathbb{Z}_2$-operations

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\tau \cdot (x_0, x_1) := (-x_0, -x_1), \quad (x_0, x_k) \cdot \tau := (-x_0, x_1).
\]

**Proposition**

\[
|\text{Hom}(K_2, C_{2r+1})| \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times S^1_b
\]
The easy part

- The composition map

$$\text{Hom}(K_2, C_{2r+1}) \times \text{Hom}(C_{2r+1}, G) \rightarrow \text{Hom}(K_2, G)$$

yields

$$S^1_b \times \mathbb{Z}_2 |\text{Hom}(C_{2r+1}, G)| \rightarrow \mathbb{Z}_2 |\text{Hom}(K_2, G)|.$$
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- Direct consequence:

\[ \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(C_{2r+1}, G)| + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|. \]
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- The composition map

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- This already proves the Lovász Conjecture.
The easy part

- The composition map
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- Direct consequence:
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- This already proves the Lovász Conjecture.
- The adjoint maps
  \[ |\text{Hom}(C_{2r+1}, G)| \longrightarrow \mathbb{Z}_2 \text{ Map}_{\mathbb{Z}_2}(S^1_b, |\text{Hom}(K_2, G)|) \]
can be fitted together to
  \[ \text{colim} |\text{Hom}(C_{2r+1}, G)| \longrightarrow \mathbb{Z}_2 \text{ Map}_{\mathbb{Z}_2}(S^1_b, |\text{Hom}(K_2, G)|). \]

math.CO/0506075.
A further hint at the relationship between \( \text{Hom}(K_2, G) \) and \( \text{Hom}(C_{2r+1}, G) \).

**Proposition**

\[
\text{Map}_{\mathbb{Z}_2}(S^1_b, \vert \text{Hom}(K_2, G) \vert) \neq \emptyset \\
\iff \vert \text{Hom}(C_{2r+1}, G) \vert \neq \emptyset \text{ for } r \text{ large enough.}
\]

**Proof.**

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**Generalizations**

joint with Babson & Dochtermann

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**Definition**

We define graphs $T_{k,r}$ for $k, r \geq 1$.

$$T_{1,r} = C_{2r+1}$$

$$T_{2,2}$$
Generalizations
joint with Babson & Dochtermann

**Definition**

We define graphs $T_{k,r}$ for $k, r \geq 1$. ($T_{1,r} = C_{2r+1}, \ldots$)

**Theorem**

$$\lim_{r \to \infty} \text{coind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \geq \text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

$$\text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$
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\[
\begin{align*}
\cohom\text{-}\mathrm{ind}_{\mathbb{Z}_2}|\Hom(T_{k,r}, G)| + k &\leq \cohom\text{-}\mathrm{ind}_{\mathbb{Z}_2}|\Hom(K_2, G)| \\
\lim_{r \to \infty} \coind_{\mathbb{Z}_2}|\Hom(T_{k,r}, G)| + k &\geq \coind_{\mathbb{Z}_2}|\Hom(K_2, G)|
\end{align*}
\]

Corollary

\[
\begin{align*}
\coind_{\mathbb{Z}_2}|\Hom(K_2, G)| &\leq \max \{ k : \text{Ex. } r \geq 1 \text{ and } T_{k,r} \to G \} \\
&\leq \cohom\text{-}\mathrm{ind}_{\mathbb{Z}_2}|\Hom(K_2, G)| \leq \chi(G) - 2.
\end{align*}
\]
## Generalizations

joint with Babson & Dochtermann

### Definition

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<table>
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| $\chi(T_{k,r}) \geq k + 2$  
| $\chi(T_{k,r}) = k + 2$, if $r$ is large enough. |
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joint with Babson & Dochtermann

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We define graphs $T_{k,r}$ for $k, r \geq 1$. ($T_{1,r} = C_{2r+1}, \ldots$)

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\text{cohom-ind}_{\mathbb{Z}_2}|\text{Hom}(T_{k,r}, G)| + k \leq \text{cohom-ind}_{\mathbb{Z}_2}|\text{Hom}(K_2, G)|
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\lim_{r \to \infty} \text{coind}_{\mathbb{Z}_2}|\text{Hom}(T_{k,r}, G)| + k \geq \text{coind}_{\mathbb{Z}_2}|\text{Hom}(K_2, G)|
\]

Corollary
- $\text{coind}_{\mathbb{Z}_2}|\text{Hom}(K_2, G)| \leq \max \{k : \text{Ex. } r \geq 1 \text{ and } T_{k,r} \to G\}$
  $\leq \text{cohom-ind}_{\mathbb{Z}_2}|\text{Hom}(K_2, G)| \leq \chi(G) - 2.$
- $\chi(T_{k,r}) \geq k + 2, \quad \chi(T_{k,r}) = k + 2, \text{ if } r \text{ is large enough.}$
- If e.g. $K$ is a Kneser graph with $\chi(K) = k + 2$, then
  $\text{coind}_{\mathbb{Z}_2}|\text{Hom}(K_2, K)| = k$ and hence there exist $r$ and $T_{k,r} \to K$. 
In the limit, the $\mathbb{Z}_2$-homotopy type of $\text{Hom}(C_{2r+1}, G)$ is determined by the $\mathbb{Z}_2$-homotopy type of $\text{Hom}(K_2, G)$.
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The bounds on $\chi(G)$ obtained from $\text{Hom}(K_2, G)$ and $\text{Hom}(C_{2r+1}, G)$ for large $r$ are essentially the same.
In the limit, the $\mathbb{Z}_2$-homotopy type of $\text{Hom}(C_{2r+1}, G)$ is determined by the $\mathbb{Z}_2$-homotopy type of $\text{Hom}(K_2, G)$.

The bounds on $\chi(G)$ obtained from $\text{Hom}(K_2, G)$ and $\text{Hom}(C_{2r+1}, G)$ for large $r$ are essentially the same.

$\text{coind}_{\mathbb{Z}_2} \text{Hom}(K_2, G)$ can be described combinatorially via the existence of graph homomorphisms to $G$. 