“Topology”
Problem Set 1

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1. Topological Spaces

1. Let $(X_i)_{i\in I}$ be a system of spaces and Y a space. Denote the projection maps by $\pi_j\colon \prod_{i\in I} X_i \to X_j$. Show that a function $f\colon Y \to \prod_{i\in I} X_i$ is continuous if and only if for all $i \in I$ the component function $\pi_i \circ f$ is continuous.

2. Let X be a topological space. Prove that the following statements are equivalent.
   (i) X is a Hausdorff space.
   (ii) The diagonal $\Delta := \{(x,x) \in X \times X\}$ is closed in $X \times X$.
   (iii) For every topological space Z, every dense subset $D \subseteq Z$ (i.e. every $D \subseteq Z$ such that $\overline{D} = Z$), and all continuous functions $f_1, f_2\colon Z \to X$, the equality $f_1|_D = f_2|_D$ implies $f_1 = f_2$.

3. Let X, Y be compact topological spaces. Show (without using Tychonoff's theorem) that $X \times Y$ is compact.

   Instructions. Consider an open covering of $X \times Y$. For every $x \in X$ show that there is a finite subcover of $\{x\} \times Y$. Apply a proposition from the exercise session (stating that the map $X \times Y \to X$ is closed) and finally the compactness of X.

4. Prove the following statements.
   (i) The unit interval is connected.
   (ii) Every path connected space is connected.

5. **Extra Problem (no credit)**
   Let $G = (V, E)$ be an infinite graph with vertex set V and edge set $E \subseteq V \times V$. Let C be a finite set, the elements of which we call colours. A C-colouring of G is a function $c\colon V \to C$. The colouring c is called proper if $c(u) \neq c(v)$ for all $u, v \in V$ with $(u, v) \in E$. Show that G admits a proper C-colouring if every finite subgraph of G admits a proper C-colouring.

   Hint. Using Tychonoff's theorem make the set of all colourings of G into a compact topological space such that for every edge the set of all colourings which do satisfy the properness condition at that edge is closed. Use a characterization of compactness involving the finite intersection property.