Topology
Problem Set 11

Prof. Günter M. Ziegler
Dr. Carsten Schultz

Version date: January 23, 2008
Issue date: January 22, 2008
Hand in date: January 30/31, 2008

Class homepage:
http://carsten.codimi.de/top0708/

11. Exact sequences

38. Let X be the abstract simplicial complex on the vertex set $\{0,1,2,3\}$ consisting of the simplices $\{0,1,2\}$, $\{1,3\}$, $\{2,3\}$, and all of their faces. Let A be the subcomplex $\{\{0\},\{1\},\{2\}\}$. Work out explicitely, giving bases for the groups and matrices for the maps, the short exact sequence of chain complexes and the long exact sequence of homology groups of the pair (X, A).

39. Find a pair (X, A) of simplicial complexes such that for the canonical map $p\colon (X, \emptyset) \to (X, A)$ and some $k \geq 0$ the map $p_*\colon H_k(X) \to H_k(X, A)$ is neither injective nor surjective. Write down the complete long exact homology sequence of the pair, including all maps.

40. Let X and Y be finite simplicial complexes. Use the Künneth theorem to show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

    Remark. When working over a field (as we may do), the tensor product of a k-dimensional and an l-dimensional vector space is a kl-dimensional vector space.

41. Let A, B be abelian groups. An *extension of A by B* is a short exact sequence of the form

$$0 \longrightarrow B \xrightarrow{\ i\ } E \xrightarrow{\ p\ } A \longrightarrow 0.$$

Two such extensions $0 \to B \xrightarrow{i} E \xrightarrow{p} A \to 0$ and $0 \to B \xrightarrow{i'} E' \xrightarrow{p'} A \to 0$ are *equivalent* if there is an isomorphism $h\colon E \to E'$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\ i\ } & E & \xrightarrow{\ p\ } & A & \longrightarrow & 0 \\ & & \| & & \cong\downarrow h & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{\ i'\ } & E' & \xrightarrow{\ p'\ } & A & \longrightarrow & 0 \end{array}$$

commutes.

Determine, up to equivalence, all extensions of $\mathbb{Z}$ by $\mathbb{Z}$ and all extensions of $\mathbb{Z}_3$ by $\mathbb{Z}$.

Hint. Easy to miss might be

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m \mapsto m \oplus [2m]} \mathbb{Z} \oplus \mathbb{Z}_3 \xrightarrow{a \oplus [b] \mapsto [a+b]} \mathbb{Z}_3 \longrightarrow 0$$

or

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m \mapsto 3m} \mathbb{Z} \xrightarrow{n \mapsto [2n]} \mathbb{Z}_3 \longrightarrow 0.$$