Topology
Problem Set 12

Prof. Günter M. Ziegler
Dr. Carsten Schultz

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Class homepage:
http://carsten.codimi.de/top0708/

12. Exact sequences and cell complexes

42. There is an isomorphism $\tilde{H}_n(\mathbb{S}^n) \cong \tilde{H}_{n-1}(\mathbb{S}^{n-1})$ given by

$$\tilde{H}_n(\mathbb{S}^n) \xrightarrow{\cong} H_n(\mathbb{S}^n, \mathbb{B}^n_-) \xleftarrow{\cong} H_n(\mathbb{B}^n_+, \mathbb{S}^{n-1}) \xrightarrow[\partial_*]{\cong} \tilde{H}_{n-1}(\mathbb{S}^{n-1}),$$

where the first two maps are induced by inclusion maps of pairs and the third one is the connecting homomorphism of the long exact sequence of the pair $(\mathbb{B}^n_+, \mathbb{S}^{n-1})$. Another such isomorphism is given by

$$\tilde{H}_n(\mathbb{S}^n) \xrightarrow{\cong} H_n(\mathbb{S}^n, \mathbb{B}^n_+) \xleftarrow{\cong} H_n(\mathbb{B}^n_-, \mathbb{S}^{n-1}) \xrightarrow[\partial_*]{\cong} \tilde{H}_{n-1}(\mathbb{S}^{n-1}).$$

Do these two isomorphisms agree?

43. Let (X, A) be a CW-pair with X consisting of A plus exactly one additional cell. Let n be the dimension of this cell and $i\colon A \to X$ the inclusion map.
   (i) Prove that the homomorphism $i_*\colon H_k(A) \to H_k(X)$ is an isomorphism for $k \neq n, n-1$, an epimorphism for $k = n-1$ and a monomorphism for $k = n$.
   (ii) Prove that either $\operatorname{rank} H_{n-1}(X) = \operatorname{rank} H_{n-1}(A) - 1$ and $\operatorname{rank} H_n(X) = \operatorname{rank} H_n(A)$, or $\operatorname{rank} H_{n-1}(X) = \operatorname{rank} H_{n-1}(A)$ and $\operatorname{rank} H_n(X) = \operatorname{rank} H_n(A) + 1$.
   (iii) Give an example where neither $H_n(X) \cong H_n(A)$ nor $H_{n-1}(X) \cong H_{n-1}(A)$.

   Remark. Since we have not fully developed singular homology, you may work under additional hypotheses like triangulability.

44. (i) Show that a function $f\colon X \to Y$ from a CW complex X to a space Y is continuous if its restriction to every closed cell is continuous.

   The space $\mathbb{S}^{n+1}$ can be obtained from the space $\mathbb{S}^n$ by attaching two $(n+1)$-cells. Continuing this process *ad infinitum,* we obtain a CW-complex $\mathbb{S}^\infty$ with two cells in every dimension. Equivalently: In the set of all sequences of real numbers, identify $\mathbb{R}^n$ with the set of all sequences with all but possibly the first n elements zero. Set $\mathbb{R}^\infty = \bigcup_{n\geq 0} \mathbb{R}^n$ and define a topology on $\mathbb{R}^\infty$ such that a set is closed if and only if its intersection with every $\mathbb{R}^n$ is closed in the euclidean topology on $\mathbb{R}^n$. Let $\mathbb{S}^\infty \subseteq \mathbb{R}^\infty$ be the subspace defined by $\mathbb{S}^\infty = \bigcup_{n\geq 0} \mathbb{S}^n$.

   (ii) Show that $\mathbb{S}^\infty$ is contractible.

   Remark. You can use that $\mathbb{S}^\infty \times I$ is a CW-complex with the 'obvious' CW decomposition with four 0-cells and six i-cells for every $i > 0$.

45. Show that $\mathbb{C}\mathrm{P}^n$ has a CW decomposition with one cell in dimension $2i$ for each $0 \leq i \leq n$ and no other cells.