“Topology”
Problem Set 2

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2. Simplicial Complexes

6. Prove the following part of Lemma 2.5:

   Let Δ be a compact simplicial complex. Then Δ consists of a finite number of simplices.

7. For $m, n \in \mathbb{N}$, the (m,n)-chessboard complex $\Delta_{m,n}$ is the abstract simplicial complex

   $$\{(S \subseteq \{1,\dots,m\} \times \{1,\dots,n\} : (r,s),(r',s') \in S \implies (r \neq r' \wedge s \neq s') \vee (r,s) = (r',s')\}.$$

   It can also be described as the matching complex of the complete bipartite graph $K_{m,n}$ or as the independence complex of the graph on $\{1,\dots,m\} \times \{1,\dots,n\}$ with (r,s) a neighbor of $(r',s') \neq (r,s)$ if and only if $r = r'$ or $s = s'$.

   Draw a figure of the chessboard complex $\Delta_{4,3}$ and show that it is homeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

8. Let V be a finite set and $K \subseteq 2^V$ an abstract simplicial complex of dimension at most k. Let $x_v \in \mathbb{R}^{2k+1}$ for $v \in V$ be a system of points such that for every $A \subseteq V$ with $|A| \leq 2k+2$ the system $(x_v)_{v\in A}$ is affinely independent. Prove that $\{\operatorname{conv}\{x_s \colon s \in S\} : S \in K\}$ is a geometric realization of K.

   Remark. Such systems always exist, for example the points can be taken on the moment curve $\{(x, x^2, \dots, x^{2k+1}) \colon x \in \mathbb{R}\}$. (Remember Vandermonde's determinant.)

9. Consider the geometric simplicial complex Δ in $\mathbb{R}^2$ consisting of the 1-simplices $\operatorname{conv}\{(0,0),(1,i)\}$ for $i \in \mathbb{N}$ and their vertices. Show that the topology of $\|\Delta\|$ differs from the subspace topology on $\bigcup \Delta$.

   Extra Problem (for those with extra motivation). Instead prove the following stronger result: In the space $\|\Delta\|$ the point $(0,0)$ does not have a countable neighborhood basis (look up the definition!). Therefore $\|\Delta\|$ is not even metrizable.