“Topology”
Problem Set 4

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4. Fundamental group

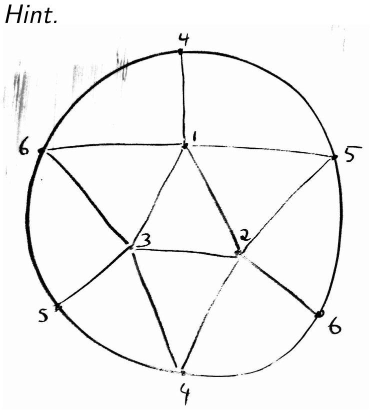
14. Show that the following three spaces are homeomorphic.
    (i) $\mathbb{S}^1 \times \mathbb{S}^1$,
    (ii) $(\mathbb{R} \times \mathbb{R})/\sim$, where $(x, y) \sim (x', y') \iff x - x' \in \mathbb{Z} \wedge y - y' \in \mathbb{Z}$,
    (iii) $(I \times I)/\sim$, where $\sim$ is the equivalence relation generated by $(r, 0) \sim (r, 1)$ and $(0, r) \sim (1, r)$ for all $r \in I$.

15. Determine $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ using the Seifert-van Kampen theorem.

    Hint. Consider the complement of a point and a small neighbourhood of that point.

16. Determine $\pi_1(\mathbb{R}\mathrm{P}^2)$ using a triangulation and Theorem 3.21.

    Hint.



17. Let $\beta\colon I \to \mathbb{S}^1 \subseteq \mathbb{C}$ be the path $\beta(t) = e^{i2\pi t}$, which represents a generator of $\pi_1(\mathbb{S}^1; 1)$.
    (i) Let $a\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the antipodal map $a(x) = -x$, let $\gamma\colon I \to \mathbb{S}^1$ be the path $\gamma(t) = -e^{-it\pi}$ from -1 to 1 and $\alpha\colon I \to \mathbb{S}^1$ any path with $\alpha(0) = 1$, $\alpha(1) = -1$. Show that the paths $\alpha * (a \circ \alpha)$ and $\alpha * \gamma * \alpha * \gamma * \beta$ represent the same element of $\pi_1(\mathbb{S}^1; 1)$.
    (ii) Let $f\colon (\mathbb{S}^1, \{1\}) \to (\mathbb{S}^1, \{1\})$ be a map with $f(-x) = -f(x)$ for all x. Show that the number determined (up to sign) by $f_\#([\beta]) \in \pi_1(\mathbb{S}^1; x_0) \cong \mathbb{Z}$ is odd.

    Hints and remarks. Since $*$ is associative only up to homotopy, the composition of more than two paths is only well defined up to homotopy. This does not pose a problem here.

    For (i) it may help to consider the map

$$H\colon I \times I \to \mathbb{S}^1$$
$$H(s,t) = \begin{cases} -e^{-it\pi}\alpha\left(\frac{3s-t}{3-2t}\right), & t \le 3s \le 3 - t, \\ -e^{-i\cdot 3s\pi}, & 3s \le t, \\ e^{i(3(s-1))\pi}, & 3s \ge 3 - t. \end{cases}$$

    Is it continuous? What are $H(0, \bullet)$, $H(1, \bullet)$, $H(\bullet, 0)$, $H(\bullet, 1)$? What does this prove?

    For (ii) note that $\beta = \gamma^- * (a \circ \gamma^-)$, where $\gamma^-(t) = \gamma(1 - t)$. Then apply (i).