

“Topology”

Problem Set 5

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5. Homotopy groups

As a definition for real projective n -space we use $\mathbb{R}P^n := \mathbb{S}^n / \sim$ with $x \sim y \iff x \in \{y, -y\}$.

18. Let us denote the unreduced suspension of a space X by ΣX and the reduced suspension by SX . Let $n \geq 0$. We have already seen that $\Sigma \mathbb{S}^n \cong \mathbb{S}^{n+1}$. Show that the canonical quotient map $\Sigma \mathbb{S}^n \rightarrow S\mathbb{S}^n$ is a homotopy equivalence.
19. Let $(X, \{x\}_0)$ be a space with a base point, $n \geq 1$, and $f: (\mathbb{S}^n, \{e_1\}) \rightarrow (X, \{x_0\})$ a continuous map. Let $g: (I^n, \partial I^n) \rightarrow (\mathbb{S}^n, \{e_1\})$ be a map which induces a homeomorphism $I^n / \partial I^n \xrightarrow{\cong} \mathbb{S}^n$ and let $u = [g] \in \pi_n(\mathbb{S}^n; x_0)$. Prove that the following statements are equivalent.
 - (i) $f_{\#}(\alpha) = e$ for all $\alpha \in \pi_n(\mathbb{S}^n; x_0)$,
 - (ii) $f_{\#}(u) = e$,
 - (iii) f is homotopic as a map of pairs to the constant map sending \mathbb{S}^n to x_0 .
 - (iv) f is freely homotopic to a constant map.
 - (v) f can be extended to a continuous map $\hat{f}: \mathbb{B}^{n+1} \rightarrow X$.

Remark. Prove the implications from top to bottom and then (v) \implies (i). For this last implication use functoriality of the homotopy groups and that we know the homotopy groups of contractible spaces. Please do not just state that (iv) \implies (i) is known from class, since one goal of this problem is to prove this implication.

20. Let $i_n: \mathbb{R}P^n \rightarrow \mathbb{R}P^{n+1}$ denote the map induced by the inclusion $\mathbb{S}^n \rightarrow \mathbb{S}^{n+1}$.
 - (i) Since $\mathbb{R}P^1 \cong \mathbb{S}^1$, we know that $\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$. In Problem 16, you have calculated $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$. Revisit this calculation to determine the homomorphism $(i_1)_{\#}: \pi_1(\mathbb{R}P^1) \rightarrow \pi_1(\mathbb{R}P^2)$.
 - (ii) Use the Seifert–van Kampen theorem to show that for $n > 1$ the homomorphism $(i_n)_{\#}: \pi_1(\mathbb{R}P^n) \rightarrow \pi_1(\mathbb{R}P^{n+1})$ is an isomorphism.