“Topology”
Problem Set 6

Prof. Günter M. Ziegler
Dr. Carsten Schultz

Version date: November 28, 2007
Issue date: November 27, 2007
Hand in date: December 5/6, 2007

Class homepage:
`http://carsten.codimi.de/top0708/`

6. Homology

21. Let Δ be an abstract simplicial complex and t a vertex of Δ. Assume that Δ is a cone with tip t, i.e. that for every simplex σ of Δ, the set $\sigma \cup \{t\}$ is also a simplex of Δ. Show that the maps

$$D_i \colon \tilde{C}_i(\Delta) \to \tilde{C}_{i+1}(\Delta)$$
$$[v_0, \dots, v_i] \mapsto \begin{cases} [t, v_0, \dots, v_i], & t \notin \{v_0, \dots, v_i\} \\ 0, & t \in \{v_0, \dots, v_i\} \end{cases}$$

are well-defined and form a chain homotopy between $\mathrm{id}_{\tilde{C}_*(\Delta)}$ and the zero map. Use this to prove that $\tilde{H}_i(\Delta) \cong 0$ for all i (this is Lemma 4.13). What is $H_i(\Delta)$?

22. Let $A_0 := \mathbb{Z}$, $A_1 := \mathbb{Z} \oplus \mathbb{Z}$, $A_2 := \mathbb{Z}$, and $A_i := 0$ for all $i \neq 0, 1, 2$. In each of the following cases, give maps $\mathfrak{d}_i \colon A_i \to A_{i-1}$ which make A into a chain complex with the desired homology groups, or show that this is not possible.
    - (i) $H_2(A) \cong 0$, $H_1(A) \cong 0$, $H_0(A) \cong 0$.
    - (ii) $H_2(A) \cong 0$, $H_1(A) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$, $H_0(A) \cong 0$.
    - (iii) $H_2(A) \cong 0$, $H_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, $H_0(A) \cong \mathbb{Z}$.
    - (iv) $H_2(A) \cong 0$, $H_1(A) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H_0(A) \cong 0$.
    - (v) $H_2(A) \cong 0$, $H_1(A) \cong \mathbb{Z}_5$, $H_0(A) \cong \mathbb{Z}_3$.
    - (vi) $H_2(A) \cong \mathbb{Z}$, $H_1(A) \cong \mathbb{Z}$, $H_0(A) \cong \mathbb{Z}_2$.
    - (vii) $H_2(A) \cong \mathbb{Z}$, $H_1(A) \cong \mathbb{Z}$, $H_0(A) \cong \mathbb{Z}$.
    - (viii) $H_2(A) \cong \mathbb{Z}$, $H_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}$, $H_0(A) \cong \mathbb{Z}$.
    - (ix) $H_2(A) \cong \mathbb{Z}$, $H_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, $H_0(A) \cong 0$.

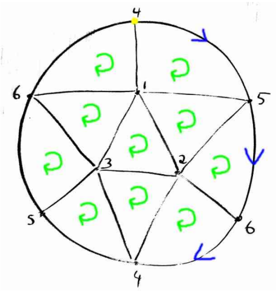
23. Let Δ be the simplicial complex from Problem 16, which triangulates $\mathbb{RP}^2$, and let G be one of the abelian groups $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_2$. We define groups $A_0, A_1, A_2 := G$ and maps $h_i \colon A_i \to C_i(\Delta; G)$ by, for $g \in G$,

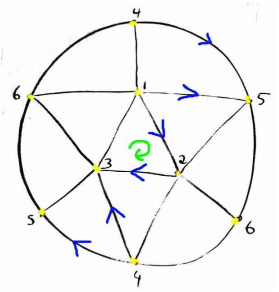
$$\begin{aligned} h_0(g) &:= g[4], \\ h_1(g) &:= g[4,5] + g[5,6] + g[6,4], \\ h_2(g) &:= g[4,5,1] + g[1,5,2] + g[2,5,6] + g[6,4,1] + g[6,1,3] + \\ &\qquad + g[3,1,2] + g[3,2,4] + g[4,2,6] + g[6,3,5] + g[5,3,4]. \end{aligned}$$

We also define maps $f_i \colon C_i(\Delta; G) \to A_i$ by setting $f_0(g[v]) := g$ for any vertex v,

$$f_1(g[4,5]) = f_1(g[1,4]) = f_1(g[1,2]) = f_1(g[2,3]) = f_1(g[4,3]) = g$$

and $f_1(g\sigma) = 0$ for all other 1-simplices σ, $f_2(g[1,2,3]) = g$ and $f_2(g\sigma) = 0$ for all other 2-simplices σ.





- (i) Show that there are maps $\mathfrak{d}_2 \colon A_2 \to A_1$ and $\mathfrak{d}_1 \colon A_1 \to A_0$ which make the groups A_i into a chain complex in such a way that the maps f_i become a chain map and also the maps h_i.
- (ii) Having made A_* into a chain complex in this way, show that the chain maps $h \circ f$ and $f \circ h$ are chain homotopic to identity maps by constructing the relevant chain homotopies.
- (iii) Use this to determine the groups $H_*(\Delta; \mathbb{Z})$, $H_*(\Delta; \mathbb{Q})$, and $H_*(\Delta; \mathbb{Z}_2)$.

Alternative version, pick one. (iii') Calculate $H_n(A_*)$ (with the boundary maps of A_* as determined in (i)) for $G = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_2$ and all n. (ii') Show that $h_* \colon H_n(A_*) \to H_n(\Delta; G)$ is an isomorphism for all n.