Topology
Problem Set 7

Prof. Günter M. Ziegler
Dr. Carsten Schultz

Version date: December 10, 2007
Issue date: December 4, 2007
Hand in date: December 12/13, 2007

Class homepage:
`http://carsten.codimi.de/top0708/`

7. Homology

24. For $k \in \mathbb{Z}$ define $f_k : \mathbb{S}^1 \to \mathbb{S}^1$ by $f_k(z) = z^k$, where we regard $\mathbb{S}^1$ as a subset of $\mathbb{C}$. Show that for all $x \in H_1(\mathbb{S}^1)$ we have $(f_k)_*(x) = kx$.

25. Let Δ be a geometric simplicial complex with vertex set V and assume a total ordering on V. Let $J \subseteq V \times V$ be the set $J := \{(u, v) : u \le v \text{ and } \operatorname{conv}\{u, v\} \in \Delta\}$ and set $x_{(u,v)} := \frac{1}{2}u + \frac{1}{2}v$ for $(u, v) \in J$. Then $V' := \{x_j : j \in J\}$ consists of all vertices of Δ and all barycenters of edges of Δ. Let Δ' be the set of simplices

$$\Delta' := \big\{ \operatorname{conv}\{x_{(u_i,v_i)} : 0 \le i \le k\} : (u_i, v_i) \in J, u_0 \le u_1 \le \cdots \le u_k \le v_k \le \cdots \le v_1 \le v_0, \operatorname{conv}\{u_0, \dots, u_k, v_0, \dots, v_k\} \in \Delta\big\}.$$

   (i) Draw a figure for the case that Δ is a 2-simplex.
   (ii) Prove (in general) that Δ' is a subdivision of Δ.

   Note. This is done for example in [Wal88].

26. For $n \ge 0$ let Σ^n be the simplicial complex in $\mathbb{R}^{n+1}$ with simplices $\operatorname{conv}\{\varepsilon_0 e_{i_0}, \dots, \varepsilon_k e_{i_k}\}$ with $\varepsilon_j \in \{+1, -1\}$ and $1 \le i_0 < i_1 < \cdots < i_k \le n+1$. We have $\|\Sigma^n\| \cong \mathbb{S}^n$.

   Let $n, m \ge 0$ and $(U_i)_{1 \le i \le m+1}$ be a system of open subsets of $\|\Sigma^n\|$ such that

$$\bigcup_{i=1}^{m+1} (U_i \cup -U_i) = \|\Sigma^n\| \qquad (1)$$

   and for every i we have

$$U_i \cap -U_i = \emptyset. \qquad (2)$$

   Show that there is a centrally symmetric subdivision Δ of Σ^n and a simplicial map $f : \Delta \to \Sigma^m$ such that

$$f(-x) = -f(x) \quad \text{for all } x \in \|\Sigma^n\| \qquad (3)$$

   and such that

$$f(v) = e_i \text{ implies } v \in U_i \qquad (4)$$

   for all vertices v of Δ.

   Hint. Because of (1) one can map the vertices of Σ^n to the vertices of Σ^m such that (3) and (4) are satisfied. However, this does not define a simplicial map in general. (Why?) Therefore, it is useful to replace (4) by a stronger condition in the spirit of the simplicial approximation theorem and follow the proof of that theorem. (Where does (2) come into play?)

   Note. We will learn later on that a map $f : \Delta \to \Sigma^m$ which satisfies (3) can only exist for $m \ge n$, this restricts the existence of set systems (U_i) with (1) and (2). We will also learn that for $m = n$ the map f will have to be surjective; together with your proof this should yield that in this case the intersection of all of the U_i has to be non-empty. (Do you see this?) Even more can be said, see [Fan52].

References

[Fan52] Ky Fan. A generalization of Tucker's combinatorial lemma with topological applications. *Annals Math.*, 56:431–437, 1952.

[Wal88] James W. Walker. Canonical homeomorphisms of posets. *European J. Combin.*, 9(2):97–107, 1988.