Topology
Problem Set 8

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8. Euler and Lefschetz numbers

27. Let Y be a normal space (Def. 1.8), J a finite set and $(X_j)_{j\in J}$ a system of subsets of Y.
    (i) Show that if all of the X_j are open and $\bigcup_{j\in J} X_j = Y$ then there exist closed sets A_j with $A_j \subseteq X_j$ for all $j \in J$ and $\bigcup_{j\in J} A_j = Y$.
    (ii) Show that if all of the X_j are closed then there exist open sets $U_j \subseteq Y$ with $X_j \subseteq U_j$ and $\bigcap_{j\in M} U_j = \emptyset$ for all $M \subseteq J$ with $\bigcap_{j\in M} X_j = \emptyset$.

    Hint. Inductive procedures will work better if you prove stronger statements. For example you might not only shrink the open sets to closed sets which cover X, but to closed sets whose interiors cover X.

    Note. This problems indicates how to generally switch between open and closed sets in Borsuk–Ulam–Lusternik–Schnirelman type statements.

28. Let Δ be a finite simplicial complex, J a finite set, and $(\Gamma_j)_{j\in J}$ a covering of Δ by subcomplexes. The *nerve* $\mathcal{N}(\Gamma)$ *of the covering* Γ is the abstract simplicial complex with vertex set J and simplices all the subsets $M \subseteq J$ such that $\bigcap_{j\in M} \|\Gamma_j\| \neq \emptyset$. Show that if $\tilde{\chi}\left(\bigcap_{j\in M} \Gamma_j\right) = 0$ for all $M \in \mathcal{N}(\Gamma)$, we have $\chi(\Delta) = \chi(\mathcal{N}(\Gamma))$.

    Hint. Induction on the cardinality of J. How are generally $\chi(A)$, $\chi(B)$, $\chi(A\cup B)$, and $\chi(A\cap B)$ related for subcomplexes A and B of some simplicial complex?

29. Let Δ be a finite simplicial complex. Show that $\chi(\mathbb{S}^1 \times \|\Delta\|) = 0$.

    Note. You may use without proof that $I \times \|\Delta\|$ can be triangulated in such a way that on $\{0\} \times \|\Delta\|$ and $\{1\} \times \|\Delta\|$ one obtains the original triangulation Δ. Sticking these together yields a triangulation of $\mathbb{S}^1 \times \Delta$, indeed many of them.

30. The linear map $(x, y) \mapsto (-x-y, x)$ induces a map f from the torus $T^2 \cong (\mathbb{R}\times\mathbb{R})/(\mathbb{Z}\times\mathbb{Z})$ (second description in problem 14) to itself.
    (i) What are the fixed points of f?
    (ii) Determine the induced map $f_*\colon H_*(T^2) \to H_*(T^2)$.
    (iii) Show that every map which is homotopic to f has a fixed point.

    Note. We consider the homology groups of T^2 to be known. In particular, if $x_0 \in \mathbb{S}^1$ and $i^0, i^1\colon \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ denote the maps $i^0(x) = (x, x_0)$, $i^1(x) = (x_0, x)$, you may use that $\mathbb{Z} \oplus \mathbb{Z} \cong H_1(\mathbb{S}^1) \oplus H_1(\mathbb{S}^1) \xrightarrow{i^0_* + i^1_*} H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ is an isomorphism.