

As promised, these are some suggestions (and just that) of problems and topics for discussion to help you consolidate your understanding of homology. I will leave the books mentioned below in Michael's office.

1. To see that cellular homology is actually a useful tool for the computation of homology:
 - (a) Continue our example of the decomposition of a sphere into two cells in every dimension: The antipodal map $a: \mathbb{S}^n \rightarrow \mathbb{S}^n$ satisfies $a^2 = \text{id}$ and $a_*(\sigma_i^+) = \pm \sigma_i^-$. (Why?) Work out the chain map a_* with just this information and use this to again determine $a_*: \tilde{H}_n(\mathbb{S}^n) \rightarrow \tilde{H}_n(\mathbb{S}^n)$. The decomposition of the sphere and the projection map $p: \mathbb{S}^n \rightarrow \mathbb{R}P^n$ induce a cell decomposition of $\mathbb{R}P^n$ with one cell τ_i in every dimension. We can assume $p_*(\sigma_i) = \tau_i$ and we of course have $p \circ a = p$. Use this to determine the boundary maps of the cellular chain complex of $\mathbb{R}P^n$ and the groups $H_i(\mathbb{R}P^n)$.
 - (b) The spaces $\mathbb{S}^2 \vee \mathbb{S}^1$ and $\mathbb{S}^2 \cup \mathbb{D}^1$ (for the latter consider \mathbb{R}^1 as a subset of \mathbb{R}^3) are homotopy equivalent. (This was a problem in last semester's homework.) Calculate their homologies using small cell decompositions (I am thinking of 3 respectively 7 cells) to see that they are actually the same. Find cellular maps in both directions which are homotopy inverse to each other, and check that they induce isomorphisms in homology. If you want to carry this one step further, also consider the homotopies that show that these two maps are homotopy inverse to each other. These maps can be chosen to be cellular and then induce chain homotopies on the level of the cellular chains. Work these out. (Here we consider, for a CW-complex X , $X \times I$ to be a CW-complex to which each i -cell of X contributes two i -cells and one $(i+1)$ -cell. This $(i+1)$ -cell has to be considered for the definition of the chain homotopy.) Without having worked this out, I would guess that the composition $\mathbb{S}^2 \vee \mathbb{S}^1 \rightarrow \mathbb{S}^2 \cup \mathbb{D}^1 \rightarrow \mathbb{S}^2 \vee \mathbb{S}^1$ already induces the identity on the cellular chain level, so that the chain homotopy would be zero, while for the other composition the chain homotopy will be nontrivial. If the signs start to bug you too much, work over \mathbb{Z}_2 . The signs are not the point here.
2. Make sure that you understand that ordered simplicial homology can be derived as a special case from cellular homology. See [Bre93, IV.20–21].
3. The following problem did not make into last year's final exam. Would it have been too difficult?

Let A be a subspace of X .

- (a) What is the long exact homology sequence of the pair (X, A) ?
- (b) Assume that A is a retract of X , i.e. there exists a map $r: X \rightarrow A$ such that $r \circ i = \text{id}_A$. Which maps or groups in this long exact sequence are necessarily zero?

- (c) Assume that A is a deformation retract of X , i.e. there exists a map $r: X \rightarrow A$ such that $r \circ i = \text{id}_A$ and $i \circ r \simeq \text{id}_X$. Which maps or groups in this long exact sequence are necessarily zero?
4. While proving that singular homology satisfies the axioms, we constructed several natural chain maps and also a natural chain homotopy. Make sure that you fully understand these constructions; several similar ones are about to follow. When saying ‘natural’ we mean that something is a natural transformation, you might want to look up a definition of this term, e.g. [Dol72, I.1.9]. If you feel more adventurous, have a look at [Dol72, VI.11] and see if that has any relevance to what we have done.
5. When proving the theorem about small simplices, we only proved that the inclusion chain map induces an isomorphism in homology, we did not produce a homotopy inverse to it. Read the full proof in [Spa66, 4.4.14].

References

- [Bre93] BREDON, G. E. *Topology and Geometry*, vol. 139 of *Graduate Texts in Mathematics*. Springer-Verlag, 1993.
- [Dol72] DOLD, A. *Lectures on Algebraic Topology*, vol. 200 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer-Verlag, 1972.
- [Spa66] SPANIER, E. *Algebraic Topology*. McGraw-Hill, New York, 1966.