The Lovász Conjecture and extensions

Graph colourings, spaces of edges and spaces of circuits

Carsten Schultz

Technische Universität Berlin

Workshop on Topological Methods in Combinatorics
Stockholm 2006

Outline

Outline

Hom-posets and -complexes

- A *graph homomorphism* $G \to H$ is a function $V(G) \to V(H)$ that preserves the adjacency relation.
- A *multi-homomorphism* $\phi\colon G \to H$ is

$$\phi\colon V(G) \to \mathcal{P}(V(H)) \setminus \{\emptyset\}$$

such that every choice function for ϕ is a homomorphism

Hom-posets and -complexes

- A *graph homomorphism* $G \to H$ is a function $V(G) \to V(H)$ that preserves the adjacency relation.
- A *multi-homomorphism* $\phi\colon G \to H$ is

$$\phi\colon V(G) \to \mathcal{P}(V(H)) \setminus \{\emptyset\}$$

such that every choice function for ϕ is a homomorphism

- $\mathrm{Hom}(G, H)$ is the *poset* of all multi-homomorphisms from G to H.
- The composition map

$$\begin{aligned} *\colon \mathrm{Hom}(G, G') \times \mathrm{Hom}(G', G'') &\longrightarrow \mathrm{Hom}(G, G'') \\ (\phi * \rho)(v) &:= \rho[\phi(v)] \end{aligned}$$

is associative and order-preserving.

Hom-posets and -complexes

- A *graph homomorphism* $G \to H$ is a function $V(G) \to V(H)$ that preserves the adjacency relation.
- A *multi-homomorphism* $\phi\colon G \to H$ is

$$\phi\colon V(G) \to \mathcal{P}(V(H)) \setminus \{\emptyset\}$$

such that every choice function for ϕ is a homomorphism

- $\mathrm{Hom}(G, H)$ is the *poset* of all multi-homomorphisms from G to H.
- The composition map

$$\begin{aligned} *\colon \mathrm{Hom}(G, G') \times \mathrm{Hom}(G', G'') &\longrightarrow \mathrm{Hom}(G, G'') \\ (\phi * \rho)(v) &:= \rho[\phi(v)] \end{aligned}$$

is associative and order-preserving.
- $\alpha\colon G \to G$ with $\alpha^2 = \mathrm{id}$ makes $|\mathrm{Hom}(G, H)|$ into a free $\mathbb{Z}_2$-space, if α flips an edge and H is loop-free.
- In particular, $|\mathrm{Hom}(K_2, H)|$ is a free $\mathbb{Z}_2$-space.

Hom-posets and -complexes

- A *graph homomorphism* $G \to H$ is a function $V(G) \to V(H)$ that preserves the adjacency relation.
- A *multi-homomorphism* $\phi\colon G \to H$ is

$$\phi\colon V(G) \to \mathcal{P}(V(H)) \setminus \{\emptyset\}$$

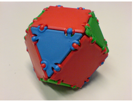
such that every choice function for ϕ is a homomorphism

- $\mathrm{Hom}(G, H)$ is the *poset* of all multi-homomorphisms from G to H.
- The composition map

$$\begin{aligned} *\colon \mathrm{Hom}(G, G') \times \mathrm{Hom}(G', G'') &\longrightarrow \mathrm{Hom}(G, G'') \\ (\phi * \rho)(v) &:= \rho[\phi(v)] \end{aligned}$$

is associative and order-preserving.

- $\alpha\colon G \to G$ with $\alpha^2 = \mathrm{id}$ makes $|\mathrm{Hom}(G, H)|$ into a free $\mathbb{Z}_2$-space, if α flips an edge and H is loop-free.
- In particular, $|\mathrm{Hom}(K_2, H)|$ is a free $\mathbb{Z}_2$-space.
- $|\mathrm{Hom}(K_2, K_n)| \approx_{\mathbb{Z}_2} \mathbb{S}^{n-2}$.



Lovász' Theorem and Conjecture

Theorem (Lovász '78)

Let G be a graph. Then

$$\chi(G) \geq \text{ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| + 2.$$

Theorem (Babson & Kozlov '04)

Let G be a graph, $r \geq 1$*. Then*

$$\chi(G) \geq \text{coind}_{\mathbb{Z}_2} |\text{Hom}(C_{2r+1}, G)| + 3.$$

Lovász' Theorem and Conjecture

Theorem (Lovász '78)

Let G be a graph. Then

$$\chi(G) \geq \mathrm{ind}_{\mathbb{Z}_2}|\mathrm{Hom}(K_2, G)| + 2.$$

Theorem (Babson & Kozlov '04)

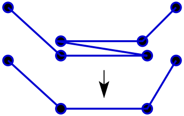
Let G be a graph, $r \geq 1$. Then

$$\chi(G) \geq \mathrm{coind}_{\mathbb{Z}_2}|\mathrm{Hom}(C_{2r+1}, G)| + 3.$$

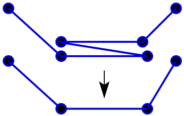
Question

What is the relationship between $\mathrm{Hom}(K_2, G)$ *and* $\mathrm{Hom}(C_{2r+1}, G)$*?*

- There is a homomorphism $C_{m+2} \to_{\mathbb{Z}_2} C_m$.
- This induces $\mathrm{Hom}(C_m, G) \to_{\mathbb{Z}_2} \mathrm{Hom}(C_{m+2}, G)$.
- We can consider $\mathrm{colim}_r |\mathrm{Hom}(C_{2r+1}, G)|$.



- There is a homomorphism $C_{m+2} \to_{\mathbb{Z}_2} C_m$.
- This induces $\mathrm{Hom}(C_m, G) \to_{\mathbb{Z}_2} \mathrm{Hom}(C_{m+2}, G)$.
- We can consider $\mathrm{colim}_r |\mathrm{Hom}(C_{2r+1}, G)|$.



Theorem

$$\operatorname*{colim}_r |\mathrm{Hom}(C_{2r+1}, G)| \simeq_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, |\mathrm{Hom}(K_2, G)|)$$
$$\operatorname*{colim}_r |\mathrm{Hom}(C_{2r}, G)| \simeq_{\mathbb{Z}_2} \mathrm{Map}(\mathbb{S}^1_b, |\mathrm{Hom}(K_2, G)|)$$

Outline

Spaces of cycles of arbitrary lengths

in complete graphs

- Since $|\mathrm{Hom}(K_2, K_n)| \approx_{\mathbb{Z}_2} \mathbb{S}^{n-2}$:
  - $\mathrm{colim}_r |\mathrm{Hom}(C_{2r}, K_n)| \simeq \mathrm{Map}(\mathbb{S}^1, \mathbb{S}^{n-2})$ free loop space of a sphere.
  - $\mathrm{colim}_r |\mathrm{Hom}(C_{2r+1}, K_n)| \simeq \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^{n-2})$.

Kozlov, D. N. Cohomology of colorings of cycles, 2005. math.AT/0507117.

- Since $|\mathrm{Hom}(K_2, K_n)| \approx_{\mathbb{Z}_2} \mathbb{S}^{n-2}$:
  - $\mathrm{colim}_r |\mathrm{Hom}(C_{2r}, K_n)| \simeq \mathrm{Map}(\mathbb{S}^1, \mathbb{S}^{n-2})$ free loop space of a sphere.
  - $\mathrm{colim}_r |\mathrm{Hom}(C_{2r+1}, K_n)| \simeq \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^{n-2})$.
- There is a canonical embedding map

$$V_{2,n-1} := \big\{(x, y) \in \mathbb{S}^{n-2} \colon \langle x, y\rangle = 0\big\} \longrightarrow \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^{n-2})$$

"Start at x, follow great circle through y."

Kozlov, D. N. Cohomology of colorings of cycles, 2005. math.AT/0507117.

Spaces of cycles of arbitrary lengths

in complete graphs

- Since $|\mathrm{Hom}(K_2, K_n)| \approx_{\mathbb{Z}_2} \mathbb{S}^{n-2}$:
  - $\mathrm{colim}_r |\mathrm{Hom}(C_{2r}, K_n)| \simeq \mathrm{Map}(\mathbb{S}^1, \mathbb{S}^{n-2})$ free loop space of a sphere.
  - $\mathrm{colim}_r |\mathrm{Hom}(C_{2r+1}, K_n)| \simeq \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^{n-2})$.
- There is a canonical embedding map

$$V_{2,n-1} := \{(x,y) \in \mathbb{S}^{n-2} \colon \langle x, y\rangle = 0\} \longrightarrow \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{S}^{n-2})$$

"Start at x, follow great circle through y."

Theorem (S, conjectured by Csorba)

$$|\mathrm{Hom}(C_5, K_n)| \approx V_{2,n-1}.$$

Kozlov, D. N. Cohomology of colorings of cycles, 2005. math.AT/0507117.

Csorba, P. *Non-tidy Spaces and Graph Colorings*. Ph.D. thesis, ETH Zürich, 2005.

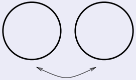
Csorba, P. and Lutz, F. H. Graph coloring manifolds, 2005. math.CO/0510177.

Spaces of cycles of arbitrary lengths

in cyclic graphs

Proposition

$$|\mathrm{Hom}(K_2, C_{2r+1})| \approx_{\mathbb{Z}_2}$$ 

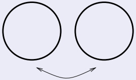
$$|\mathrm{Hom}(K_2, C_{2r})| \approx_{\mathbb{Z}_2}$$ 

Spaces of cycles of arbitrary lengths

in cyclic graphs

Proposition

$$|\mathrm{Hom}(K_2, C_{2r+1})| \approx_{\mathbb{Z}_2}$$ 

$$|\mathrm{Hom}(K_2, C_{2r})| \approx_{\mathbb{Z}_2}$$ 

Corollary

$$\operatorname*{colim}_r |\mathrm{Hom}(C_{2r}, C_m)| \simeq \mathrm{Map}(\mathbb{S}^1, |\mathrm{Hom}(K_2, C_m)|) \simeq \coprod_{\mathbb{Z}} \mathbb{S}^1$$

$$\operatorname*{colim}_r |\mathrm{Hom}(C_{2r+1}, C_m)| \simeq \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1, |\mathrm{Hom}(K_2, C_m)|) \simeq \begin{cases} \coprod_{\mathbb{Z}} \mathbb{S}^1, & m \text{ odd}, \\ \emptyset, & m \text{ even}. \end{cases}$$

Čukić, S. L. and Kozlov, D. N. The homotopy type of complexes of graph homomorphisms between cycles. *Discrete Comp. Geometry*. In press, math.CO/0408015.

Free $\mathbb{Z}_2$-spaces

Definition

Let X be a free $\mathbb{Z}_2$-space. We define

$$\operatorname{ind}_{\mathbb{Z}_2} X := \min\left\{k\colon \text{There is a } \mathbb{Z}_2\text{-map } X \to \mathbb{S}^k\right\},$$

$$\operatorname{coind}_{\mathbb{Z}_2} X := \max\left\{k\colon \text{There is a } \mathbb{Z}_2\text{-map } \mathbb{S}^k \to X\right\},$$

$$\operatorname{cohom-ind}_{\mathbb{Z}_2} X := \max\left\{k\colon f^*(\gamma^k) \neq 0\right\},$$

where $f\colon X/\mathbb{Z}_2 \to \mathbb{R}\mathrm{P}^\infty$ is classifying and $H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[\gamma]$.

Free $\mathbb{Z}_2$-spaces

Definition

Let X be a free $\mathbb{Z}_2$-space. We define

$$\text{ind}_{\mathbb{Z}_2} X := \min \left\{ k \colon \text{There is a } \mathbb{Z}_2\text{-map } X \to \mathbb{S}^k \right\},$$

$$\text{coind}_{\mathbb{Z}_2} X := \max \left\{ k \colon \text{There is a } \mathbb{Z}_2\text{-map } \mathbb{S}^k \to X \right\},$$

$$\text{cohom-ind}_{\mathbb{Z}_2} X := \max \left\{ k \colon f^*(\gamma^k) \neq 0 \right\},$$

where $f \colon X/\mathbb{Z}_2 \to \mathbb{R}\mathrm{P}^\infty$ is classifying and $H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[\gamma]$.

Properties

- $\text{conn}\, X + 1 \leq \text{coind}_{\mathbb{Z}_2} X \leq \text{cohom-ind}_{\mathbb{Z}_2} X \leq \text{ind}_{\mathbb{Z}_2} X$
- $\text{coind}_{\mathbb{Z}_2} \mathbb{S}^n = \text{ind}_{\mathbb{Z}_2} \mathbb{S}^n = n$
- *If there is* $X \to_{\mathbb{Z}_2} Y$ *then* $\text{x-ind}_{\mathbb{Z}_2} X \leq \text{x-ind}_{\mathbb{Z}_2} Y$.

Some topological facts

Let X, Y be free $\mathbb{Z}_2$-spaces and $\mathbb{S}^1_b$ the 1-sphere with the $\mathbb{Z}_2$-operations

$$\tau \cdot (x_0, x_1) := (-x_0, -x_1), \qquad (x_0, x_k) \cdot \tau := (-x_0, x_1).$$

- There is an adjunction

$$\mathfrak{Top}^{\mathbb{Z}_2}(Y, \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, X)) \cong \mathfrak{Top}^{\mathbb{Z}_2}(\mathbb{S}^1_b \times_{\mathbb{Z}_2} Y, X).$$

Some topological facts

Let X, Y be free $\mathbb{Z}_2$-spaces and $\mathbb{S}^1_b$ the 1-sphere with the $\mathbb{Z}_2$-operations

$$\tau \cdot (x_0, x_1) := (-x_0, -x_1), \qquad (x_0, x_k) \cdot \tau := (-x_0, x_1).$$

- There is an adjunction

$$\mathfrak{Top}^{\mathbb{Z}_2}(Y, \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, X)) \cong \mathfrak{Top}^{\mathbb{Z}_2}(\mathbb{S}^1_b \times_{\mathbb{Z}_2} Y, X).$$

- We obtain inequalities

$$\begin{aligned}
\operatorname{cohom-ind}_{\mathbb{Z}_2} X + 1 &\leq \operatorname{cohom-ind}_{\mathbb{Z}_2}(\mathbb{S}^1_b \times_{\mathbb{Z}_2} X), \\
\operatorname{ind}_{\mathbb{Z}_2}(\mathbb{S}^1_b \times_{\mathbb{Z}_2} X) &\leq \operatorname{ind}_{\mathbb{Z}_2} X + 1, \\
\operatorname{cohom-ind}_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, X) + 1 &\leq \operatorname{cohom-ind}_{\mathbb{Z}_2} X, \\
\operatorname{coind}_{\mathbb{Z}_2} X &\leq \operatorname{coind}_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, X) + 1.
\end{aligned}$$

Theorems (Lovász '78, Babson & Kozlov '04)

$$\chi(G) \geq \mathrm{ind}_{\mathbb{Z}_2}|\mathrm{Hom}(K_2, G)| + 2.$$

$$\chi(G) \geq \mathrm{coind}_{\mathbb{Z}_2}|\mathrm{Hom}(C_{2r+1}, G)| + 3.$$

Theorem

$$\mathrm{colim}_r|\mathrm{Hom}(C_{2r+1}, G)| \simeq_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, |\mathrm{Hom}(K_2, G)|).$$

Theorems (Lovász '78, Babson & Kozlov '04)

$$\chi(G) \geq \operatorname{ind}_{\mathbb{Z}_2} |\operatorname{Hom}(K_2, G)| + 2.$$
$$\chi(G) \geq \operatorname{coind}_{\mathbb{Z}_2} |\operatorname{Hom}(C_{2r+1}, G)| + 3.$$

Theorem

$$\operatorname{colim}_r |\operatorname{Hom}(C_{2r+1}, G)| \simeq_{\mathbb{Z}_2} \operatorname{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, |\operatorname{Hom}(K_2, G)|).$$

Corollary

$$\operatorname{cohom-ind}_{\mathbb{Z}_2} |\operatorname{Hom}(C_{2r+1}, G)| + 1 \leq \operatorname{cohom-ind}_{\mathbb{Z}_2} |\operatorname{Hom}(K_2, G)|.$$
$$\lim_{r\to\infty} \operatorname{coind}_{\mathbb{Z}_2} |\operatorname{Hom}(C_{2r+1}, G)| + 1 \geq \operatorname{coind}_{\mathbb{Z}_2} |\operatorname{Hom}(K_2, G)|.$$

Outline

$\mathrm{Hom}(K_2, C_{2r+1})$

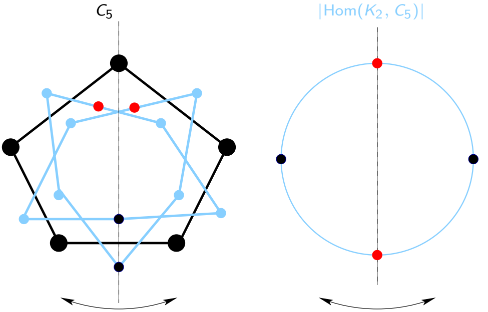
A closer look.

Reminder

We want to compare $\mathrm{Hom}(K_2, G)$ *and* $\mathrm{Hom}(C_{2r+1}, G)$*.*

$\text{Hom}(K_2, C_{2r+1})$

A closer look.



Definition

$\mathbb{S}^1_b$ is the 1-sphere with the $\mathbb{Z}_2$-operations

$$\tau \cdot (x_0, x_1) := (-x_0, -x_1),$$
$$(x_0, x_k) \cdot \tau := (-x_0, x_1).$$

Proposition

$$|\text{Hom}(K_2, C_{2r+1})| \approx_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mathbb{S}^1_b$$

The easy part

- The composition map

$$\mathrm{Hom}(K_2, C_{2r+1}) \times \mathrm{Hom}(C_{2r+1}, G) \longrightarrow \mathrm{Hom}(K_2, G)$$

yields

$$\mathbb{S}^1_b \times_{\mathbb{Z}_2} |\mathrm{Hom}(C_{2r+1}, G)| \longrightarrow_{\mathbb{Z}_2} |\mathrm{Hom}(K_2, G)|.$$

The easy part

- The composition map

$$\mathrm{Hom}(K_2, C_{2r+1}) \times \mathrm{Hom}(C_{2r+1}, G) \longrightarrow \mathrm{Hom}(K_2, G)$$

yields

$$\mathbb{S}^1_b \times_{\mathbb{Z}_2} |\mathrm{Hom}(C_{2r+1}, G)| \longrightarrow_{\mathbb{Z}_2} |\mathrm{Hom}(K_2, G)|.$$

- Direct consequence:

$$\text{cohom-ind}_{\mathbb{Z}_2}|\mathrm{Hom}(C_{2r+1}, G)| + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2}|\mathrm{Hom}(K_2, G)|.$$

The easy part

- The composition map

$$\mathrm{Hom}(K_2, C_{2r+1}) \times \mathrm{Hom}(C_{2r+1}, G) \longrightarrow \mathrm{Hom}(K_2, G)$$

  yields

$$\mathbb{S}^1_b \times_{\mathbb{Z}_2} |\mathrm{Hom}(C_{2r+1}, G)| \longrightarrow_{\mathbb{Z}_2} |\mathrm{Hom}(K_2, G)|.$$

- Direct consequence:

$$\text{cohom-ind}_{\mathbb{Z}_2}|\mathrm{Hom}(C_{2r+1}, G)| + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2}|\mathrm{Hom}(K_2, G)|.$$

- This already proves the Lovász Conjecture.

Živaljević, R. T. Parallel transport of Hom-complexes and the Lovász conjecture, 2005. math.CO/0506075.

The easy part

- The composition map

$$\mathrm{Hom}(K_2, C_{2r+1}) \times \mathrm{Hom}(C_{2r+1}, G) \longrightarrow \mathrm{Hom}(K_2, G)$$

  yields

$$\mathbb{S}^1_b \times_{\mathbb{Z}_2} |\mathrm{Hom}(C_{2r+1}, G)| \longrightarrow_{\mathbb{Z}_2} |\mathrm{Hom}(K_2, G)|.$$

- Direct consequence:

$$\text{cohom-ind}_{\mathbb{Z}_2}|\mathrm{Hom}(C_{2r+1}, G)| + 1 \le \text{cohom-ind}_{\mathbb{Z}_2}|\mathrm{Hom}(K_2, G)|.$$

- This already proves the Lovász Conjecture.
- The adjoint maps

$$|\mathrm{Hom}(C_{2r+1}, G)| \longrightarrow_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, |\mathrm{Hom}(K_2, G)|)$$

  can be fitted together to

$$\operatorname*{colim}_r |\mathrm{Hom}(C_{2r+1}, G)| \longrightarrow_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, |\mathrm{Hom}(K_2, G)|).$$

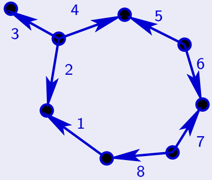
Živaljević, R. T. Parallel transport of Hom-complexes and the Lovász conjecture, 2005. math.CO/0506075.

A further hint at the relationship between $\mathrm{Hom}(K_2, G)$ and $\mathrm{Hom}(C_{2r+1}, G)$.

Proposition

$\mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, |\mathrm{Hom}(K_2, G)|) \neq \emptyset$
$\iff |\mathrm{Hom}(C_{2r+1}, G)| \neq \emptyset$ *for r large enough.*

Proof.



□

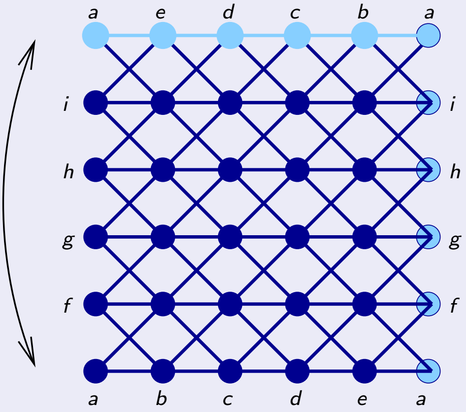
LOVÁSZ, L. Kneser's conjecture, chromatic number and homotopy. *J. Combinatorial Theory, Ser. A*, **25**:319–324, 1978.

Outline

Definition

We define graphs $T_{k,r}$ for $k, r \geq 1$.

$$T_{1,r} = C_{2r+1}$$



$T_{2,2}$

Generalizations

joint with Babson & Dochtermann

Definition

We define graphs $T_{k,r}$ for $k, r \geq 1$. $\quad (T_{1,r} = C_{2r+1}, \ldots)$

Theorem

$$\text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

$$\lim_{r\to\infty} \text{coind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \geq \text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

Generalizations

joint with Babson & Dochtermann

Definition

We define graphs $T_{k,r}$ for $k, r \geq 1$. $\quad (T_{1,r} = C_{2r+1}, \ldots)$

Theorem

$$\text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

$$\lim_{r \to \infty} \text{coind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \geq \text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

Corollary

- $\text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| \leq \max \{k \colon \textit{Ex. } r \geq 1 \textit{ and } T_{k,r} \to G\}$
  $\leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| \leq \chi(G) - 2.$

Generalizations

joint with Babson & Dochtermann

Definition

We define graphs $T_{k,r}$ for $k, r \geq 1$. $\quad (T_{1,r} = C_{2r+1}, \ldots)$

Theorem

$$\text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

$$\lim_{r \to \infty} \text{coind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \geq \text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

Corollary

- $\text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| \leq \max\{k \colon \textit{Ex. } r \geq 1 \textit{ and } T_{k,r} \to G\}$
  $\leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| \leq \chi(G) - 2.$
- $\chi(T_{k,r}) \geq k + 2, \quad \chi(T_{k,r}) = k + 2$, *if r is large enough.*

Generalizations

joint with Babson & Dochtermann

Definition

We define graphs $T_{k,r}$ for $k, r \geq 1$. $\quad (T_{1,r} = C_{2r+1}, \ldots)$

Theorem

$$\text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

$$\lim_{r \to \infty} \text{coind}_{\mathbb{Z}_2} |\text{Hom}(T_{k,r}, G)| + k \geq \text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)|$$

Corollary

- $\text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| \leq \max\{k \colon \textit{Ex. } r \geq 1 \textit{ and } T_{k,r} \to G\}$
  $\leq \text{cohom-ind}_{\mathbb{Z}_2} |\text{Hom}(K_2, G)| \leq \chi(G) - 2.$
- $\chi(T_{k,r}) \geq k + 2, \quad \chi(T_{k,r}) = k + 2$, *if* r *is large enough.*
- *If e.g.* K *is a Kneser graph with* $\chi(K) = k + 2$, *then* $\text{coind}_{\mathbb{Z}_2} |\text{Hom}(K_2, K)| = k$ *and hence there exist* r *and* $T_{k,r} \to K$.

Summary

- In the limit, the $\mathbb{Z}_2$-homotopy type of $\mathrm{Hom}(C_{2r+1}, G)$ is determined by the $\mathbb{Z}_2$-homotopy type of $\mathrm{Hom}(K_2, G)$.

Summary

- In the limit, the $\mathbb{Z}_2$-homotopy type of $\mathrm{Hom}(C_{2r+1}, G)$ is determined by the $\mathbb{Z}_2$-homotopy type of $\mathrm{Hom}(K_2, G)$.
- The bounds on $\chi(G)$ obtained from $\mathrm{Hom}(K_2, G)$ and $\mathrm{Hom}(C_{2r+1}, G)$ for large r are essentially the same.

Summary

- In the limit, the $\mathbb{Z}_2$-homotopy type of $\mathrm{Hom}(C_{2r+1}, G)$ is determined by the $\mathbb{Z}_2$-homotopy type of $\mathrm{Hom}(K_2, G)$.
- The bounds on $\chi(G)$ obtained from $\mathrm{Hom}(K_2, G)$ and $\mathrm{Hom}(C_{2r+1}, G)$ for large r are essentially the same.
- $\mathrm{coind}_{\mathbb{Z}_2} \mathrm{Hom}(K_2, G)$ can be described combinatorially via the existence of graph homomorphisms to G.